Signal Processing and Reconstruction: FIDs to Images

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Introduction

One of the intriguing features of MRI is that we image a 2D vector object: the projection of a 3D magnetization vector onto a plane perpendicular to the main magnetic field.

This 2D vector object is conveniently described using complex numbers, and hence the MRI signal processing and reconstruction can best be discussed in terms related to the manipulation of complex valued data.

In this lecture we present the mathematics behind MR signal generation, spatial encoding, data acquisition, signal demodulation, and image reconstruction.

Complex Valued MR Signals

The key towards understanding the MR image generation process involves developing an appreciation for the complex valued nature of the signal.

The combined nuclear spins of all the hydrogen protons within a voxel can be treated as a 3D magnetization vector which, in its equilibrium state, is aligned with the main magnetic field, defined here as being z-directed. If this 3D vector is knocked out of this equilibrium state (typically by the application of suitable RF energy) it precesses about its z-axis, at a frequency γB_0, where γ is the gyromagnetic ratio and B_0 is the main magnetic field strength.

Magnetic induction, the process by which a signal is induced in an MR receive coil, requires a (fairly rapidly) changing signal. Now the only component of the 3D vector changing rapidly enough is that perpendicular to the z-axis – in other words that component rotating in, what we denote as, the x-y plane. This is the component responsible for the so-called free induction decay (FID) that is detected by the MR receiver.

MR, then, detects a 2D magnetization vector resulting from the projection of a 3D magnetization vector onto the x-y plane. We can write this time varying 2D magnetization vector as

$B_{xy}(t) = \rho(\hat{x}\cos(\omega_0 t + \vartheta) + \hat{y}\sin(\omega_0 t + \vartheta))$;

where ρ is the magnitude of the projection onto the x-y plane, $\omega_0 = \gamma B_0$, the precession frequency, $\hat{x}$ and $\hat{y}$ are unit vectors in the x and y directions respectively, and ϑ defines the orientation of the vector in the x-y plane.

For convenience (mainly to simplify the notation) we replace our 2D vector with a complex number, whereby the real and imaginary parts corresponds to the x and y directed components, respectively. Hence we can also write

$B_{xy}(t) = \rho \exp(i(\omega_0 t + \vartheta))$.

The signal we detect at the receive coil depends on B_{xy} and the complex (valued) coil sensitivity. Further, ρ depends on the tissue's proton density, its T1 and T2 relaxation rates, the complex transmit coil sensitivity, and the pulse sequence. For simplicity's sake we will assume that the coil sensitivities, tissue, and pulse sequence impacts, and phase at $t = 0$, are all included in a new complex term ρ. Hence we have

$B_{xy}(t) = \rho(t)\exp(i\omega_0 t) = |\rho(t)|\exp(i\varphi(t))\exp(i\omega_0 t)$.

Recovering the Complex MR Signal

Now the signal in the coil is necessarily real valued, or

$s(t) = Re\{B_{xy}\} = |\rho(t)|\cos(\omega_0 t + \varphi(t))$.

To create an image we need to be able to recover the complex $\rho(t)$; and the way this is achieved is via complex signal demodulation. Basically this involves separately multiplying the signal by both a cosine and a sine term, and combining the two resultant signals as real and imaginary parts of a synthesized complex signal.

Hence we start with

$s_{cd}(t) = |\rho(t)| \cos(\omega_0 t + \varphi(t)) \left(\cos(\omega_0 t) - isin(\omega_0 t)\right);$

then after some algebra, and ignoring high frequency terms (those with $2\omega_0 t$), we obtain

$s_{cd}(t) = |\rho(t)|(\cos(\varphi(t)) + isin(\varphi(t)) = |\rho(t)| \exp(i\varphi(t)) = \rho(t).$

Note that ω_0 is a fixed (reference) frequency; and the high frequency terms are typically removed with low pass filters.

In phasor signal notation, $\rho(t)$ is known as the complex amplitude of the signal $B_{xy}(t)$. In MRI $\rho(t)$ is also often known as the rotating frame signal.

The MR receiver, then, is designed to provide us with this complex magnetization signal.

Manipulating the Complex Magnetization

One nice aspect of the complex notation is that rotations of a 2D vector can be performed by multiplication with a complex exponential. Further, all gradient induced manipulations of the magnetization, as used for spatial encoding, are simply 2D rotations in the x-y plane.

For frequency encoding, the rotation is time varying, and the receiver signal is given by

$s(t) = \rho \exp\,(i\gamma G_x t x);$

where G_x is the strength of the gradient field, and x is the x-position of the voxel.

Note that the time variation on the ρ term is usually only due to relaxation, and can typically be ignored without much error.

For phase encoding, each view is acquired with a different rotation. Hence in the presence of phase encoding the signal is given by

$s(t,n) = \rho \exp\,(i\gamma n g_y T_p y);$

where g_y is the incremental step in the phase encoding gradient amplitude, T_p is the length of the phase encoding pulse, y is the y-position of the voxel, and n is typically an integer incrementing from -(N/2) to (N/2)-1.

Notice that phase encoding requires us to introduce a 2D data structure for our signal.

The Full Signal

The MR signal equation is linear. Hence the full signal is just the integral of that from each phase and frequency encoded voxel over the 2D x-y plane (we'll assume a slice selective acquisition), and is given by

$S(t,n) = \iint \rho(x,y) \exp\left(i\gamma\left(G_x t x + n g_y T_p y\right)\right) dxdy.$

K-Space

The k-space representation is obtained by writing the full signal equation with a change of variables. Define two spatial frequency coordinates $\{u, v\} = \{\gamma G_x t, \gamma n g_y T_p\}$; then

$S_k(u,v) = \iint \rho(x,y) \exp\left(i(ux + vy)\right)\ dxdy.$

The beauty of this expression is that it clearly describes the 2D Fourier Transform relationship between the (2D) signal, $S_k(u,v)$, and the spatially dependent magnetization, $\rho(x,y)$.

Image Reconstruction

The above Fourier relationship implies that the spatial magnetization distribution can be recover by inverse Fourier transforming the (2D) signal. Or

$\rho(x,y) = \iint S_k(u,v) \exp\,(-i(ux + vy)\,)dudv;$

which is a very well conditioned operation, and can typically be rapidly computed using the Fast Fourier Transform algorithm.