Minimum-noise Laplacian kernel for MR-based electrical properties tomography

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Target Audience: MR scientist with interest in MR-based tissue electrical property measurement.

Introduction: MREPT, in its most widely used form, utilizes the Laplacian of the measured RF field map (say B_1^+) to calculate the relative permittivity (ϵ_r) and the electrical conductivity (σ) of tissue in an ROI (Eqs. (1,2)). Noise in ϵ_r and σ (electrical properties, EP) is often dominated by the noise of the Laplacian of the B_1^+ map. Different methods exist in literature to calculate $\nabla^2 B_1^+$ with robustness to noise [1-4], but quantitative comparison among different methods is often missing. Here we show that among all linear Laplacian kernels—a 3D matrix which gives a Laplacian estimator when multiplied by a data matrix—the one based on the Savitzky-Golay 2nd order derivative kernel has the least amount of noise amplification. Numerical simulation verifies the theoretical results.

$$\nabla^2 B_1^+ = -k^2 B_1^+ \qquad \text{Eq. (1)}$$

$$k^2 = \omega^2 \epsilon_r \epsilon_0 \mu_0 - i\omega\sigma\mu_0 \qquad \text{Eq. (2)}$$

$$\boldsymbol{g} = (0\;0\;0\;0\;0\;0\;0\;2\;2\;2)(\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T \qquad \text{Eq. (3)}$$

$$\boldsymbol{F} \equiv \left(\boldsymbol{1}^T, \boldsymbol{x}^T, \boldsymbol{y}^T, \boldsymbol{z}^T, (\boldsymbol{xy})^T, (\boldsymbol{yz})^T, (\boldsymbol{zx})^T, \left(\boldsymbol{x^2}\right)^T, \left(\boldsymbol{y^2}\right)^T, \left(\boldsymbol{z^2}\right)^T\right) \qquad \text{Eq. (4)}$$

$$\boldsymbol{g} = 2\boldsymbol{a}_{x^2}^{SG} + 2\boldsymbol{a}_{y^2}^{SG} + 2\boldsymbol{a}_{z^2}^{SG} \qquad \text{Eq. (5)}$$

Figure 1. Equations. $\omega = 2\pi f$ is the MRI frequency. ϵ_0 and μ_0 are the permittivity and the permeability of vacuum, respectively.

Theory: Suppose that a row vector $\boldsymbol{b}$: $b_v, v = 1,2,3,\cdots, N_{tot}$ contains a 3D complex RF field map in an ROI consisting of N_{tot} voxels. A Laplacian kernel can be defined as a row vector $\boldsymbol{g}$: g_v, $v \in$ ROI that estimates the Laplacian in the ROI when projected to b_v: $estimator(\nabla^2 b) = \langle b \rangle_g \equiv \boldsymbol{b}\cdot\boldsymbol{g}$. For $\langle b \rangle_g$ to be a good Laplacian estimator, it has to be zero when b_v is constant or a linear function of the voxel coordinates, and produces an exact Laplacian when b_v is a purely parabolic function of the coordinates. We impose, therefore, ten conditions on $\boldsymbol{g}$: $\boldsymbol{g}\cdot\boldsymbol{1} = \boldsymbol{g}\cdot\boldsymbol{x} = \boldsymbol{g}\cdot\boldsymbol{y} = \boldsymbol{g}\cdot\boldsymbol{z} = \boldsymbol{g}\cdot\boldsymbol{xy} = \boldsymbol{g}\cdot\boldsymbol{yz} = \boldsymbol{g}\cdot\boldsymbol{zx} = 0$, and $\boldsymbol{g}\cdot\boldsymbol{x^2} = \boldsymbol{g}\cdot\boldsymbol{y^2} = \boldsymbol{g}\cdot\boldsymbol{z^2} = 2$. Here, $\boldsymbol{1}, \boldsymbol{x}, \boldsymbol{y}, \cdots$ are row vectors representing polynomial terms of the Cartesian coordinates of each voxel in the ROI. For a given (voxel-independent) noise δb in the input b_v, the noise in the output $\langle b \rangle_g$ is proportional to the root sum of the squares of the kernel g_v: $\delta\langle b \rangle_g = \delta(\boldsymbol{b}\cdot\boldsymbol{g}) = \delta b \cdot \sqrt{\boldsymbol{g}\cdot\boldsymbol{g}}$, provided the noise in b_v at different v is statistically independent. Finding $\boldsymbol{g}$ to minimize $\boldsymbol{g}\cdot\boldsymbol{g}$ subject to the ten conditions above is an analytically solvable problem. Using Lagrange's multipliers, the solution can be found as Eqs. (3,4). It turns out that Eq. (3) is exactly twice the sum of the Savitzky-Golay second-order derivative kernels [5] $\boldsymbol{a}_{x^2}^{SG}, \boldsymbol{a}_{y^2}^{SG}, \boldsymbol{a}_{z^2}^{SG}$ along the three Cartesian directions in the same ROI (Eq. (5)). By the definition of the Savitzky-Golay filter [6], Eq. (5) gives the Laplacian obtainable by 3D least-squares quadratic fitting of the input data b_v. Therefore, the Laplacian estimators obtained by the minimum-noise Laplacian kernel, by the Savitzky-Golay 2nd-order derivative kernels, and by 3D quadratic fitting, are *all the same*.

Assign Cartesian coordinates to each of the N_{tot} voxels in the ROI. The origin of the coordinates can be assigned to a point in the approximate center of the ROI.

↓

Form a $10 \times N_{tot}$ matrix $\boldsymbol{F}$ out of column vectors of unity ($\boldsymbol{1}$), quadratic ($\boldsymbol{x^2}, \boldsymbol{y^2}, \boldsymbol{z^2}$), linear ($\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$) and cross ($\boldsymbol{xy}$, $\boldsymbol{yz}, \boldsymbol{zx}$) terms.

↓

Calculate $h \equiv (\boldsymbol{F}^T\boldsymbol{F})^{-1}\boldsymbol{F}^T$.

↓

The Laplacian kernel is given by twice the sum of the three rows of h that correspond to $\boldsymbol{x^2}, \boldsymbol{y^2}, \boldsymbol{z^2}$.

↓

The final answer is given by reshaping the Laplacian kernel into the shape of the ROI.

Figure 2. Recipe to construct a minimum-noise Laplacian kernel for an arbitrary ROI.

Methods: Publicly available Matlab codes for multi-dimensional Savitzky-Golay differentiation kernel [7] can be readily adapted to rapidly compute the minimum-noise Laplacian of the RF field in an arbitrary ROI. The concrete steps are shown in Fig. 2. In order to demonstrate the benefit of the proposed kernel, we compared the SNR of the EP maps obtained with different Laplacian kernels applied to a synthetic B_1^+ map. The test B_1^+ map was obtained by the analytical RF solution [8] on an infinitely long cylinder with diameter 20 cm and ϵ_r = 80, σ = 0.8 S/m at 128 MHz. Independent Gaussian random noise was added to both the magnitude and the phase of the B_1^+ map so that $SNR_{|B_1^+|} = SNR_{\angle B_1^+} = 200$ at the center of the cylinder. Three Laplacian kernels defined on the ROI shown in Fig. 3(b) were compared: (1) A nearest neighbor (nn) Laplacian kernel, defined by averaging voxel-wise nearest neighbor Laplacian on all eligible (away from the boundary) voxels in the ROI. (2) van Lier's (vL) kernel used in ref. [2], and the Savitzky-Golay (SG) kernel described above. Maps of ϵ_r and σ were obtained on an axial slice of the cylinder by computing ϵ_r and σ according to Eqs. (1,2) as the ROI was swept across the slice.

Results: Figure 3(a) shows that the EP maps computed with the SG kernel visually has the least noise. Figure 3(c) shows the EP statistics, confirming higher SNR obtained by the SG kernel.

Discussion: Although nonlinear method [4] was not considered here, a linear Laplacian kernel includes many published methods [1-3] of Laplacian computation. The finding of a best-case linear method can help systematic comparison between linear and nonlinear methods by providing a reference. Such reference can also help analyze the relative importance of random vs systematic noise in the accuracy of different MREPT methods.

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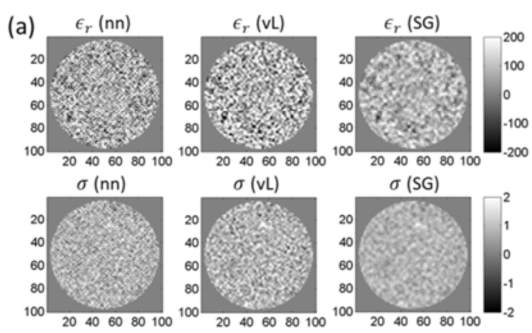

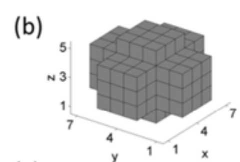

(c)

kernel	nn	vL	SG
ϵ_r	78.7 ± 133	78.7 ± 90.3	78.6 ± 52.9
σ [S/m]	0.802 ± 0.720	0.800 ± 0.497	0.797 ± 0.286

Figure 3. (a) Simulated MREPT results using 3 different Laplacian kernels. (b) Common shape of the kernel (ROI). Voxel size =2 mm. (c) Electrical properties statistics for the maps in (a), mean ± stdev.

References: [1] Bulumulla et al., ISMRM 2012, p.3469. [2] van Lier et al., MRM 67:552-561 (2012). [3] Katscher et al., ISMRM 2012, p.3482. [4] Michel et al., ISMRM 2014, p.3192. [5] Zuo et al, Optics Express 21:5346-5362 (2011). [6] http://research.microsoft.com/en-us/um/people/jckrumm/SavGol/SavGol.htm. [7] http://www.mathworks.com/matlabcentral/fileexchange/4270-2-d-savitzky-golay-smoothing-filter. [8] Glover et al., JMR 64:255-270 (1985).