Polyhedral Phantom Framework with Analytical Fourier Transform with Intensity Gradients

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Target Audience: Scientists and physicists that perform realistic MRI simulations using physiologically relevant shapes.

Purpose: To demonstrate a computationally efficient method for simulations using analytical polyhedral phantoms with linear changes in intensity.

Introduction: Analytical phantoms with closed form solutions for their Fourier Transforms (FT) enable accurate and arbitrary sampling of *k*-space. Though most phantoms are limited to simple shapes such as rectangles or ellipsoids,[1,2] we have recently presented an expression for the 3D FT of a polyhedron of uniform intensity,[3] which was derived from the field of x-ray crystallography.[4] This phantom enables the simulation of physiologically relevant shapes.[3] One limitation of the current polyhedral analytical phantom is that it does not account for spatially varying gradients in intensity. The most intuitive method to solve this problem is to partition the polyhedron into a mesh of smaller polyhedra as would be done for finite element simulation. Each smaller shape would have different intensity, and would be Fourier-transformed independently, with the final *k*-space data obtained by adding all separate contributions. However, since these polyhedrons share some of the same polygonal faces, it would be more efficient to remove any duplicate calculation of the contribution to the final FT of each polygonal face. This work provides the mathematical framework for efficient calculation of the FT of an object with linearly changing intensity.

$$(1)\ S_{3D}(\boldsymbol{k}) = \begin{cases} -\dfrac{1}{(2\pi k))^2}\displaystyle\sum_{f=1}^{F} S_f^*(\boldsymbol{k}) & \boldsymbol{k} \neq \boldsymbol{0} \\ V & \boldsymbol{k} = \boldsymbol{0} \end{cases};\qquad (2)\ S_f^*(\boldsymbol{k} \neq k\widehat{\boldsymbol{N}}_f) = \frac{\boldsymbol{k}\cdot\widehat{\boldsymbol{N}}_f}{k^2-(\boldsymbol{k}\cdot\widehat{\boldsymbol{N}}_f)^2}\sum_{e=1}^{E_f} L_{f,e}\boldsymbol{k}\cdot\widehat{\boldsymbol{n}}_{f,e}\mathrm{sinc}(\pi\boldsymbol{k}\cdot\hat{\boldsymbol{t}}_{f,e}L_{f,e})\exp(-2\pi i\boldsymbol{k}\cdot\boldsymbol{r}^{C_{f,e}})$$

$$(3)\ S_f^*(\boldsymbol{k} = k\widehat{\boldsymbol{N}}_f) = -2\pi i\boldsymbol{k}\cdot\widehat{\boldsymbol{N}}_f\exp(-2\pi i\boldsymbol{k}\cdot\boldsymbol{r}^{v_{f,1}})P_f;\qquad (4)\ A_f = \frac{1}{2}\left|\widehat{\boldsymbol{N}}_f\sum_{e=1}^{E_f}(\boldsymbol{r}^{v_{f,e}}\times\boldsymbol{r}^{v_{f,e+1}})\right|;$$

$$(5)\ S_{3D}(\boldsymbol{k}=\boldsymbol{0}) = V = \frac{1}{6}\left|\sum_{f=1}^{F}(\boldsymbol{r}^{v_{f,1}}\cdot\widehat{\boldsymbol{N}}_f)\left|\widehat{\boldsymbol{N}}_f\cdot\sum_{e=1}^{E_f}(\boldsymbol{r}^{v_{f,e}}\times\boldsymbol{r}^{v_{f,e+1}})\right|\right|$$

Equation Set 1. Analytical Fourier Transform of a polyhedron S_{3D}(*k*. The total is arrived by summation of contributions of each face.

$$(1')\ S_{3D}(\boldsymbol{k}) = \sum_{f=1}^{F}\Delta d_f\cdot S_f^*(\boldsymbol{k});\qquad (2')\ S_f^* = \begin{cases} -\dfrac{1}{(2\pi k)^2}\cdot\dfrac{\boldsymbol{k}\cdot\widehat{\boldsymbol{N}}_f}{k^2-(\boldsymbol{k}\cdot\widehat{\boldsymbol{N}}_f)^2}\sum_{e=1}^{E_f}L_{f,e}\boldsymbol{k}\cdot\widehat{\boldsymbol{n}}_{f,e}\mathrm{sinc}(\pi\boldsymbol{k}\cdot\hat{\boldsymbol{t}}_{f,e}L_{f,e})\exp(-2\pi i\boldsymbol{k}\cdot\boldsymbol{r}^{C_{f,e}}) & \boldsymbol{k}\neq k\widehat{\boldsymbol{N}}_f \\ \dfrac{i}{2\pi k^2}\boldsymbol{k}\cdot\widehat{\boldsymbol{N}}_f\exp(-2\pi i\boldsymbol{k}\cdot\boldsymbol{r}^{v_{f,1}})A_f & \boldsymbol{k}=k\widehat{\boldsymbol{N}}_f, \boldsymbol{k}\neq\boldsymbol{0} \\ \dfrac{1}{3}A_f\boldsymbol{r}^{v_{f,1}}\cdot\widehat{\boldsymbol{N}}_f & \boldsymbol{k}=\boldsymbol{0} \end{cases}$$

Equation Set 2. Modified analytical Fourier Transform of a polyhedron with varying linear intensity.

Theory: The analytical expression for the FT of a polyhedron $S_{3D}(\boldsymbol{k})$ at a given point in k-space $\boldsymbol{k}$,[3,4] are shown in **Equation Set 1**. Let F: total faces, $\widehat{\boldsymbol{N}}_f$: unit normal of face *f*, $L_{f,e}$: length of e^{th} edge of f^{th} face, E_f: total edges in f^{th} face, $\boldsymbol{r}^{(C_{f,e})}$: position vector of the midpoint of e^{th} edge on f^{th} face, $\hat{\boldsymbol{t}}_{f,e}$: unit vector in the direction of e^{th} edge of f^{th} face, $L_{f,e}$: length of e^{th} edge of f^{th} face, $\widehat{\boldsymbol{n}}_{f,e}$: outward normal to e^{th} edge of f^{th} face in the plane of f^{th} face, $\boldsymbol{r}^{(v_{f,e})}$: position vector of e^{th} vertex of f^{th} face ordered counterclockwise when viewed against $\widehat{\boldsymbol{N}}_f$, k: magnitude of $\boldsymbol{k}$, A_f: area of the f^{th} face; V: the volume of the polyhedron. Edge *e* on face *f* points from vertex $\boldsymbol{r}^{(v_{f,e})}$ to $\boldsymbol{r}^{(v_{f,e+1})}$. Additionally, the first and last vertices are connected, thus $v_{E+1} = v_1$ and likewise $v_0 = v_E$. The modified expression that enables the efficient FT of meshes associated by changes in intensity can be seen in **Equation Set 2**, where Δd_f is the intensity change at face *f*, and all other quantities match those in Eq. Set 1. For simulation of a linear gradient in intensity across the object, the original shape is partitioned into smaller polyhedra by intersecting the original shape with planes perpendicular to the direction of intensity variation. These planes, spaced at regular intervals that depend on the chosen number of steps or partitions, create polygonal faces that are shared between neighboring polyhedra. In the proposed method, the shared faces are treated as the basic unit that contributes to the final *k*-space value, instead of repeatedly and independently computing the FT of each polyhedron weighted by a different intensity. Thus, unnecessarily duplicate calculation is avoided. By assigning intensity change Δd_f to every face, this method can deal with analytical simulation of a 3D object with changing intensity efficiently. Faces on the outside of the polyhedron are assigned Δd_f to have a value equal to the intensity within the local shape (a change from intensity 0 outside). Notice that to ensure $S_{3D}(\boldsymbol{k}=\boldsymbol{0}) > 0$, $\{\widehat{\boldsymbol{N}}_f\}_{f=1}^{F}$ should point outwards.

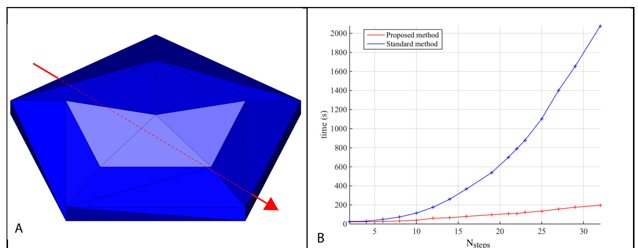

Figure 1. A: The polyhedron used for validation is composed of 40 triangular faces. The linear gradient in intensity was chosen to run the direction of the arrow. **B:** Comparison of computational time between proposed (red) and standard (blue) methods. The more efficient implementation leads to significant reduction in computation time. Efficiency is gained by calculating the contribution of each shared polygonal face only once (Equations 2 and 2').

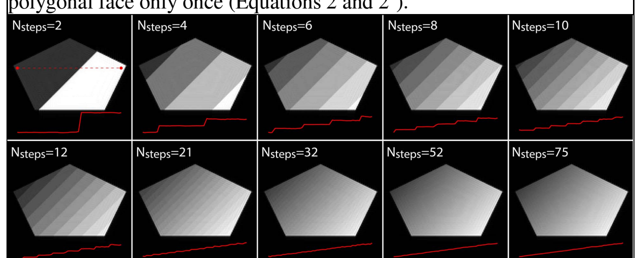

Figure 2: Cross-sectional image reconstructions of the polyhedron used for validation with increasing numbers of steps in the discretized linear intensity gradient. After 52 steps, a smooth appearance is achieved. The only limitation in number of steps is computation time.

Methods: A polyhedron composed of 40 polygonal faces was used for validation (Fig 1, A). *k*-space matrices of 64^3 or 128^3 points were used in simulations. A discretized linear gradient in intensity was with increasing number of steps ranging from 2 to 75. Computation times for implementations of both original method applied to a multiplicity of polyhedral resulting from partitioning the polyhedron and the proposed method were averaged across 3 trials (MATLAB 2014b, 2.7 GHz Intel Core i7 laptop, 16 GB RAM) ranging up to 32 steps.

Results: A visually smooth variation in intensity was generated by discretizing a linear gradient in intensity across >50 steps. The proposed calculation, which takes advantage of polygonal faces shared between neighboring polygons, was faster than the original proposed method, which calculates an FT for each individual polyhedron. Computation on our platform took 0.1516 and 0.1566 us/face/k-point for the standard proposed methods.

Conclusion: This work provides and demonstrates an efficient method for calculation of the FT of polyhedra with changing intensity. Efficient simulation of smoothly varying intensities should make simulation of physiologically relevant shapes easier.

References: [1] Shepp, Logan IEEE Trans on Nuclear Science 1974 NS-21(3): 21; [2] C. G. Koay, et al. MRM 2007, 58(2):430. [3] J. Komrska, Optik 1988, 80(4): 171; [4] T.M. Ngo, et al. ISMRM 2011.

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