Smallest Singular Value: a metric for assessing k-space sampling patterns

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Intro: When accelerating MRI acquisitions via k-space under-sampling, the choice of acquisition lines can have a large impact on the quality of the resulting images by affecting the conditioning of the image reconstruction inverse problem. Assessing the quality of a given k-space sampling pattern is typically done via the associated g-factor map[1], which gives information on the relative spatial noise amplification after reconstruction. Direct calculation of g-factor is possible[2], but computationally prohibitive for all but the simplest structured sampling grids which produce sparse, banded aliasing matrices[1]. Monte-carlo-like approaches (as the multiple-pseudo-replica [3]) are one of the best methods for generating g-factor maps of general patterns. This requires running many (typically several thousand) separate image reconstructions, each with additional random noise added, and analyzing the resultant noise distribution. This method works well, but requires image data and is still computationally difficult. We propose a new metric for assessing sampling: the smallest singular value (SSV).

Purpose: This work introduces a new, easily calculable metric for assessing k-space sampling pattern performance for parallel imaging. A different approach than g-factor is taken, where we effectively assess the conditioning of the image reconstruction problem as a whole, instead of focusing on the spatial behaviour. Generically, writing the SENSE[1] problem as a linear system, we model the measurement data y_i of the underlying magnetization x_o generated from the i^{th} coil as: $y_i = M_i\, x_o$. Here the measurement operator M_i incorporates the effects of multiplication with the receive sensitivity of coil i, followed by the Fourier transform, followed by selective sampling of k-space at locations k: $M_i = k \cdot FT \cdot S_i$. The inverse problem is then to determine x_o given the measurements from all coils, the corresponding sensitivities S, and the k-space sampling pattern k. It is clear that the performance of inverse image reconstruction will depend on the conditioning of the linear operator M. We propose to asses the maximum (m) and smallest (l for lowest) *magnitude* singular values of M: σ_m and σ_l (that is, largest magnitude, and closest to zero). Particularly important for solving the under-sampled imaging problem is σ_l, which has the physical interpretation of the worst-case overall noise amplification[4]. Systems with larger σ_l should perform better.

Methods: Calculation of the SSV metric can be carried out in a straightforward and computationally efficient manner as follows. M is generally not square, so we form the general equations $M^H M$ (where ^{H} denotes conjugate transpose), which are square and positive semi-definite, and assess the largest and smallest eigenvalues via solving the eigenvalue problem $(M^H M + R)\, v = \lambda v$ (including optional regularization term R) using ARPACK. Note that the system M or $M^H M$ are never explicitly constructed, instead it is efficient to express them as matrix-vector operations:

$$M^H M = \sum_i^{N_{coils}} S_i^H \cdot iFT \cdot k^{-1} \cdot k \cdot FT \cdot S_i$$

This formulation is very similar to that used in iterative reconstructions, minus the fit-to-data term. The eigenvalues of this system are then the square of the singular values of M, $\lambda = \sigma^2$. Importantly, unlike g-factor, computation of the SSV does not require images.

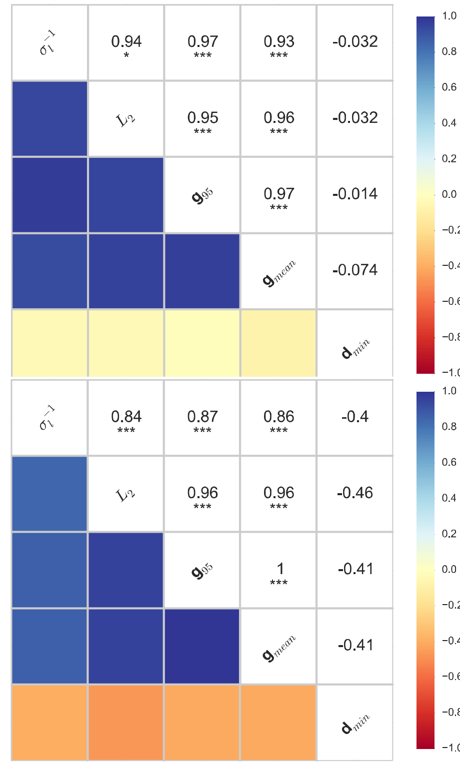

Figure 1: Rank correlations between SSV and other standard metrics (see text) for 2D-CAPI patters of R=8 (*top*) and R=16 (*bottom*).

Methods: Comparisons: To compare the validity of the SSV metric, we calculated several typical quality metrics over a family of sampling pattern grids generated via 2D-CAPI[5], used to fill a 128x128 grid of candidate k-space sampling locations for reduction factors of 8 and 16. This is the typical use case for 3D imaging, where the k-space phase/slice encode locations form a 2D array. Sensitivity maps were measured from a 31-channel head array coil at 7T[6]. For all 2D-CAPI patterns at R8 (n=15) and R16 (n=31), several metrics were computed to assess the quality of each sampling pattern. In addition to the proposed SSV metric, four other metrics were calculated for each pattern: minimum aliasing distance[5] was calculated, MPR g-factor metrics (95th percentile and mean) were computed (using 10k iterations of MPR for each), and L2-error of the resulting reconstruction. Spearman rank-correlations were measured to assess the correspondence of SSV with the reference metrics.

Methods: Utility: Having such a fast metric to assess candidate sampling patterns allows many interesting applications like on-line assessment of under-sampling patterns, and opens the door to optimization techniques for pattern selection and receive coil design. We demonstrate one simple application by looking to random sampling patterns: is there a performance difference between two commonly used random pattern generators: the uniform random distribution and the Poisson-disk. To do so, 10,000 uniform and Poisson-disk-distributed random patterns were generated over a range of reduction factors from R 1.5 – 13, and the SSV was computed for each.

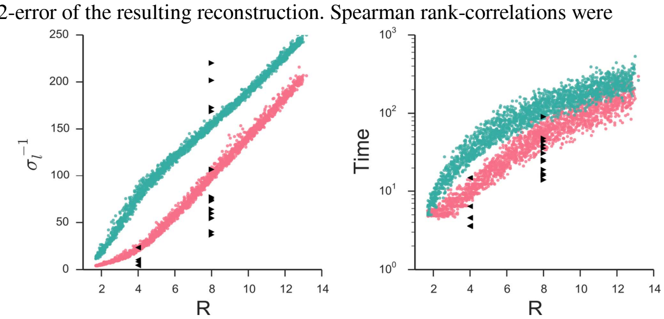

Figure 2: (*left*) reciprocal smallest singular value and (*right*) computation time for 10,000 each of Poisson-disk (*pink*) and uniformly distributed (*green*) sampling patterns. Black arrows indicate matching 2D-CAPI grids (see text) for R4 (<) and R8 (>).

Results: **Ranking & Performance**: Figure 1 demonstrates rank correlation results comparing the SSV metric with other common metrics. Note we examine the reciprocal of σ_l in order to match the usual ordering where higher g-factor is worse. SSV ranks candidate patterns very similarly to mean and 95th percentile g-factor, with most rank differences existing between very closely performing patterns, whereas patterns which are overall much worse or much better than average are ranked similarly by both g and SSV. SSV however does lose spatial information – we know if a pattern is better or worse than another, but not which regions in the image will suffer the most. SSV calculation took between several seconds, to several minutes for the worst patterns (e.g. R1x16 instead of R4x4), which have smaller σ_l and worse convergence rates for ARPACK. Computation time for multi-replica g-factor maps was orders of magnitude longer on the same computer, taking several hours per pattern.

Results: Application: Figure 2 demonstrates one example of the utility of the SSV: investigation uniform versus Poisson-disk random distributions for sampling patterns. A significant performance difference is observed between uniform and Poisson-disk patterns over the range of reductions factors, likely due to the evenly-separated nature of sample points in the Poisson disk strategy. This has important implications for planning sampling for incoherent aliasing reconstructions. Interestingly, for applications that do not require incoherent artifacts, it is easily observed that the best 2dCAPI grids far outperform random patterns, likely due to the regular aliasing producing a more structured imaging operator.

References: [1] Pruessmann, KP et al. 1999. MRM 42(5):952-62 [2] Liu, B et al, Proc ISMRM 2008 #1285 [3] Robson, PM. et al, 2008 MRM 60(4): 895-907 [4] Anand, CK et al. 2009 JMR 63-70 [5] Breuer, FA et al. 2006 MRM 55(3):549-56. [6] Gilbert KS. et al 2012 MRM 67(5): 1487-1496