What is the Lorentz sphere correction for the MRI measured field generated by tissue magnetic susceptibility: the spatial exclusivity of source and observer and the Cauchy principal value

Yi Wang[1], Dong Zhou[1], and Pascal Spincemaille[1]
[1]*Cornell University, New York, New York, United States*

Targeted audience: All attendees interested in magnetic susceptibility
Purpose: The Lorentz sphere correction is used to calculate the tissue susceptibility induced field observed by water protons in MRI[1]. The use of this imaginary sphere or its generalized version has led to unsettling discussions[2-4]: why should the specific observer geometry be involved in deriving a fundamental formula for susceptibility field?

The tissue susceptibility medium, characterized by its magnetization, and the MRI voxel signal are both macroscopic terms. The susceptibility field experienced by a specific observing nucleus is a microscopic term. We review the connections among these microscopic and macroscopic terms specific for MRI. We demonstrate that when the dipole convolution expression for the observed field generated by tissue susceptibility requires no specific observer geometry, but only the spatial exclusivity of the observer nuclei and the susceptibility source.
Method: The tissue magnetization concept is defined on a spatial smoothing of the microscopic electromagnetic field over a macroscopic scale that is much larger than the molecular scale[5]. The spatial smoothing described here defines the macroscopic field and magnetization of tissue as steady susceptibility sources and observers.

The concept of susceptibility is originally a microscopic term. For a magnetized molecule with its electron cloud polarized by the main field $\boldsymbol{B}_0$ of the MRI system, the field generated by the electron cloud polarization is called the chemical shift shielding field inside the molecule and the magnetic susceptibility field outside the molecule (modeled as the field of a magnetic moment) [6-8] (Fig.1). The spatial smoothing described in the above causes the "susceptibility field" and the "chemical-shift shielding field" to be mixed in the space outside but nearing a macromolecules. When there is very little signal detected from this mixing space (possibly caused by measurement time much longer than the effective T2), the concept of susceptibility can be translated into and preserved as a macroscopic term, and the tissue susceptibility medium with an underlying "magnetization porosity" constituting of separate source and observer spaces (Fig.2).

To describe the magnetic field generated by tissue susceptibility χ, we need to review the mathematical description of electromagnetism. All terms in the following descriptions are macroscopic terms defined by the spatial smoothing described in the above. The steady susceptibility sources are current sources of the dipole form, which is not a simple function, but a distribution (defined in the sense of integration with a test function)[9]. The magnetic field generated by a dipole located at the coordinate origin is the following distribution ($p.v.$ indicating Cauchy principal value), $\boldsymbol{b}(\boldsymbol{r}) = p.v.\left(\frac{\mu_0}{4\pi}\frac{3(\boldsymbol{\mu}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}-\boldsymbol{\mu}}{r^3}\right) + \frac{2\mu_0}{3}\boldsymbol{\mu}\delta(\boldsymbol{r})$. For an observer at a given location, the total observed field is a summation of all magnetic dipole sources, which can be represented by a dipole density or magnetization $\boldsymbol{m}(\boldsymbol{r}) \approx \chi(\boldsymbol{r})\boldsymbol{B}_0/\mu_0$. By the very definition of magnetic susceptibility, the observer is always outside the susceptibility source. Therefore, the delta distribution term in the above Eq.4 does not contribute to the summation of all dipoles or integration over magnetization in space, and the total observed field is $\boldsymbol{b}_l(\boldsymbol{r})$ (with the subscript l in honor of Lorentz)

$$\boldsymbol{b}_l(\boldsymbol{r}) = p.v.\left(\frac{\mu_0}{4\pi}\int \frac{3(\boldsymbol{m}(\boldsymbol{r}')\cdot\boldsymbol{e}_{rr'})\boldsymbol{e}_{rr'}-\boldsymbol{m}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,d^3\boldsymbol{r}'\right), \qquad [1]$$

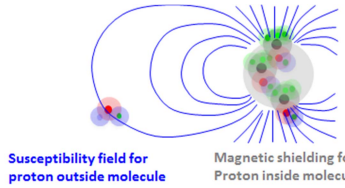

Fig.1. susceptibility is observed outside source molecule; chemical shift is observed inside source molecule.

where $\boldsymbol{e}_{rr'} = (\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'| \equiv \boldsymbol{R}/R$. Here, the Cauchy principal value integration over tissue magnetization is independent of observer geometry; it only requires that the sources and observers do not occupy the same space (spatial exclusivity) according to the definition of tissue molecular susceptibility (Fig.1). The Cauchy principal value formulation in Eq.1 forms the foundation of the forward problem from susceptibility to field in QSM.
Results: The Cauchy principal value is well defined, allowing Fourier transformation $\boldsymbol{B}(\boldsymbol{k}) = \mathcal{F}\{\boldsymbol{b}_l\}$ and differential Laplacian operation. Eq.1 becomes, respectively

$$\boldsymbol{B}(\boldsymbol{k}) = \mu_0\left(\frac{\boldsymbol{M}(\boldsymbol{k})}{3} - \hat{\boldsymbol{k}}\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{M}(\boldsymbol{k})\right)\right). \qquad [2];$$

$$\nabla^2\boldsymbol{b}_l = \mu_0\left(\frac{1}{3}\nabla^2\boldsymbol{m} - \nabla(\nabla\cdot\boldsymbol{m})\right). \qquad [3]$$

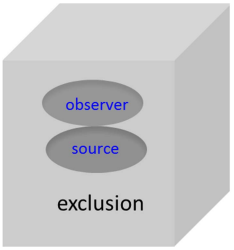

Fig.2. tissue susceptibility source and observer.

The Fourier transform of Eq.3 is Eq.2, and both are equivalent to Eq.1.

In the absence of spatial exclusivity of the source and observer, for example when the observer is an electron that is in quantum mechanical exchange with the source electrons, the apparent total field $\boldsymbol{b}$ is

$$\boldsymbol{b} = \boldsymbol{b}_l + \frac{2\mu_0}{3}\boldsymbol{m}. \qquad [4]$$

here $\frac{2\mu_0}{3}\boldsymbol{m}$ happens to be the field inside a small sphere with uniform magnetization $\boldsymbol{m}$. The difference between $\boldsymbol{b}$, the apparent field without source-observer exclusivity, and $\boldsymbol{b}_l$, the susceptibility field, can also be derived from the differential form of Maxwell's Equations, which leads to $-\nabla^2\boldsymbol{b} = \mu_0\left(-\nabla^2\boldsymbol{m} + \nabla(\nabla\cdot\boldsymbol{m})\right)$ differing from Eq.3 by $\frac{2\mu_0}{3}\nabla^2\boldsymbol{m}$.

For an ellipsoid distribution of source with susceptibility $\chi \ll 1$, the observed field is[10],

$$\boldsymbol{b}_l = \left(\frac{1}{3} - N\right)\chi\boldsymbol{B}_0, \qquad [5]$$

Discussion: The spatial exclusivity of source-observer leads to the drop of the delta distribution term in Eq.1. This was noted in an alternative derivation of the Clausius-Mossotti equation that does not depend on the shape of the envelope or cavity for the observer [11,12]. The discovery of this early work confirms our approach. Our work here provides an explicit formulation of the spatial exclusivity of the source-observer using the rigorous distribution description. Our results here provide a theoretical foundation for the forward problem in quantitative susceptibility mapping.

The spatial exclusivity affecting the susceptibility field calculation may be regarded as bearing some similarity to Pauli's principle of exclusion that affects the calculation of wave function of fermions in quantum mechanics. For a medium with the source-observer spatial exclusivity, there is less space to deposit the susceptibility source compared to a medium with the same magnetization but without the spatial exclusivity. Consequently, the observers are effectively further away from the sources, resulting in the difference for the field measured by the observed, even though both media have the same magnetization (Fig.2).
Conclusion: The spatial exclusivity of source-observer defines the magnetic susceptibility of a tissue, and the susceptibility induced field can be formulated using Cauchy principal value without referring the specific observer geometry.
References: 1. Durrant et al, *Concepts Magn Reso A* **18A**, 72-95, 10067 (2003). 2. He X & Yablonskiy D, PNAS, *A* **106**, 13558-13563 (2009). 3. Duyn J & Barbara T, MRM 72 (2014). 4. Yablonskiy et al, MRM 72 (2014). 5. Jackson, Classical Electrodynamics, 1999. 6. Landau & Lifshitz, Quantum mechanics, 1981. 7. Wang, Quantitative susceptibility mapping, 2013. 8. Duyn J, J Magn Reson 229, 198-207 (2013). 9. Stakgold, Boundary value problems of mathematical physics, 2000. 10. Landau & Lifshitz, Electrodynamics of continuous media, 1984. 11. Darwin CG, Proc Roy Soc A146, 17-46 (1934). 12. Hannay J, Eur J Phys 4, 141-143 (1983).