# Compressed Sensing

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## HIGHLIGHTS

- Compressed sensing is an exciting and rapidly growing field, attracting many attentions from different disciplines.
- Compressed sensing promises to significantly reduce the MRI scan time
- This course covers "what, why, how and where" of compressed sensing in MRI

## TARGET AUDIENCE

Scientists and clinicians interested in accelerating MRI using compressed sensing

## OUTCOME/OBJECTIVES

To understand

– What is compressed sensing

– Why we need compressed sensing in MRI

– How compressed sensing has been used in MRI

– Where compressed sensing goes in future

## PURPOSE

Compressed sensing is concerned with recovering signals from very few measurements based on the sparse representations of the signals. It has rapidly grown into a field that attracts many attentions from different disciplines. Among these disciplines, MRI is most promising to translate the compressed sensing theory into practice because MRI highly demands for reduced acquisition time and the physics of MRI naturally meets the compressed sensing requirements. The research on compressed sensing for highly accelerated MRI has exploded in the past few years. This course will teach the basics of compressed sensing and its applications in MRI.

## METHODS

Proposed by Donoho [1] and Candès et al. [2], compressed sensing started as a new sensing/sampling theory and has now become a research field that includes development in theory, algorithms, and various applications. In compressed sensing, we deal with the problem of recovering a signal (or image) $f(x)$ from very few samples or measurements

$$y_k = \langle f, \varphi_k \rangle, \ k = 1, \ldots, m \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and $\varphi_k(x)$ denotes the encoding scheme. For example, in conventional MRI, Fourier encoding is used with $\varphi_k(x) = \exp(j2\pi kx)$ being the Fourier basis. When the measurements are far fewer than what the Shannon sampling theory requires, the problem of recovering $f(x)$ from $y_k$ becomes ill-posed, suggesting there is no unique solution. With compressed sensing, the signal $f(x)$ can be recovered exactly. There are three basic components in compressed sensing [3]:

(1) The signal to be recovered is sparse or sparse after certain transformations, which means there are very few non-zeros in the unknown signal or transformed signal, but the non-zero locations are not known a priori. Mathematically, the signal $f(x)$ can be represented as:

$$f(x) = \sum_{i=1}^{n} \alpha_i \psi_i(x) \,, \tag{2}$$

where $\psi_i(x)$, $i = 1, \ldots, n$ are basis of the sparsifying transformation such that the vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ has only S(<<n) non-zero elements (called S-sparse).

(2) The encoding scheme is incoherent with the sparsifying transformation. Mathematically, the coherence is defined as $\mu = \max_{i,k} < \varphi_k, \psi_i >$ ( $1 \le \mu \le \sqrt{N}$ ), which measures the largest correlation between any pair of $\varphi_k(x)$ and $\psi_i(x)$. Intuitively, incoherent measurement ($\mu$ = 1) suggests that the signal energy spreads out in the measurement space.

(3) The signal is reconstructed using a nonlinear method enforcing both sparsity and data consistency.

In general, fewer samples are required when the signal is sparser after transformation, the coherence is lower, or the reconstruction algorithm is more rigid.

Since the MRI acquisition time is directly related to the number of samples, the application of compressed sensing to MRI for reduced acquisition time is of great interest. Such application becomes possible because MRI has the above three components of compressed sensing [4].

(1) MR images: all MR images are compressible (i.e., setting all but few largest elements after transformations to zero causes negligible perceptual loss). Figure 1 shows how sparse a brain MR image could become with finite difference or wavelet transform.

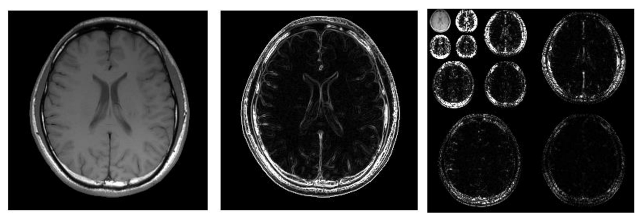

Figure 1. The brain image (left) is compressible with finite difference (middle) and wavelet transform (right).

(2) MR encoding: the MR physics allows incoherent encoding of the image. The conventional Fourier encoding in MRI is known to have maximal incoherence with the identity transform and fine scale of wavelet transform [3]. To measure the coherence of a particular k-space sampling trajectory with respect to a particular sparsifying transform, the transform point spread function [4] has been used for design of the undersampling trajectory.

(3) Reconstruction algorithm: MR images can be reconstructed within reasonable time from the incoherently undersampled k-space data using nonlinear reconstruction algorithms such as a constrained $\ell_1$ minimization algorithm [1,2]:

$$\min_{\boldsymbol{\alpha} \in R^n} \|\boldsymbol{\alpha}\|_1 \quad \text{subject to} \quad \mathbf{y} = \boldsymbol{\Phi}\boldsymbol{\Psi}^T\boldsymbol{\alpha} \qquad ( \|\boldsymbol{\alpha}\|_1 := \sum_{i=1}^{n} |\alpha_i| ) \,. \tag{3}$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ have their columns from $\varphi_k(x)$ and $\psi_i(x)$ respectively.

## EXAMPLE

Figure 2 shows a classical example of compressed sensing using the Shepp-Logan phantom. The phantom is sparse in total variation. With only 10% of the Fourier samples required by the Shannon sampling theory, nonlinear method can reconstruct the phantom image exactly, but the conventional linear method generates artifacts.

## DISCUSSION

Some of recent developments in compressed sensing MRI include sparser representations for static (e.g., [5]) or dynamic MR images (e.g., [6]), integration with parallel imaging (e.g., [7, 8]), improved optimization algorithms (e.g. [9]), non-Fourier encoding (e.g., [10, 11]), incorporation of additional prior information (e.g., [12]), and various clinical applications (e.g., [13,14]). More research is still needed for performance characterization and evaluation in clinical applications.

Figure 2. The Shepp-Logan phantom can be reconstructed from few k-space samples (left) exactly using nonlinear method (middle), while the conventional linear method fails (right).

## CONCLUSION

Compressed sensing promises to significantly reduce the MRI scan time. However, there are still challenges to be addressed in translating compressed sensing into clinical practice.

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