The Reciprocity Principle in NMR Reception

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The presentation of this material assumes familiarity with electromagnetic theory at the level of an intermediate course in fields and waves, and a knowledge of basic NMR phenomena, implied by the ability to manually solve Bloch's equations in the rotating frame. Some acquaintance with circuit theory and NMR probes is also desirable.

The reciprocity principle is a theory of linear media, and yet, for any large tip-angle of the sample moment, the NMR response is non-linear. Nonetheless, reciprocity provides our most complete picture of signal formation in NMR (*1*). Another complication is that reciprocity is *not* obeyed in gyrotropic media (*2*), i.e. those containing ferritic material, which category includes nuclear spins. Finally, it is to be emphasized, despite the near universal use of rotating coordinates in NMR, that reciprocity calculations are done in the laboratory frame; and the fact that the relevant electromagnetic quantities are written without the exponential time factor, and therefore appear stationary in time, does *not* signify that they have been transformed to rotating coordinates.

The simplest examples of reciprocity, in both reciprocal and gyrotropic media (*2*), are given in terms of the theory of two-port devices. For a passive two-port (i.e. containing no semiconductor components or power sources, and comprising only linear media), the forward and reverse transfer impedances, Z_{21} and Z_{12} are equal. More explicitly, we may drive port #1 with a voltage source, and measure the resulting short-circuit current at port # 2; we obtain the same result if the positions of drive and measurement are reversed. A more modern example gives the equality of the S parameters, S_{21} and S_{12} for forward and reverse gain.

In gyrotropic media, the situation is complicated by the presumed presence of a polarizing static field, e.g. the familiar B_0 in NMR. Then a modified form of reciprocity applies to 'non-reciprical' two-port devices, e.g. containing ferrite components. That is, the forward and reverse transfer impedances are unequal, unless the direction of the polarizing field is reversed when the positions of drive and measurement are swapped, in which case the two measured transfer impedances are equal (*2*).

While the presence of nuclear spins (say in water) has no effect upon reciprocity in circuit measurements, a simple illustration can be given for NMR. If a quadrature probe, loaded with an aqueous sample, is attached to a scanner, and an image recorded, swapping the probe cables for transmission and reception will then result in a null signal. The signal can, however, be restored by the awkward expedient of ramping the magnet down to zero field, and then re-energizing it with reversed polarity.
The discussion is made more quantitative by introduction of the magnetic susceptibility χ, which gives the magnetization in linear media in terms of the magnetic field as $\mathbf{M} = \boldsymbol{\chi}\mathbf{H}$. In isotropic phases, χ is usually a scalar quantity, although in many solids χ is anisotropic and given by a tensor; it may also be defined for static or oscillatory magnetic fields, and the early literature of NMR is filled with discussions of the AC magnetic susceptibility, considered as a scalar.

In fact, for NMR, χ has the unusual property that it is a tensor quantity , even in isotropic phases, such as liquids or gases. For nuclei of positive gyromagnetic ratio, we write (*3*):

$$\boldsymbol{\chi} = \{\chi'(\omega) + i\chi''(\omega)\} \begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [1]$$

where the complex tensor is multiplied by the complex scalar susceptibility, whose (typically Lorentzian) form does not concern us at the moment. This tensor obeys the gyrotropic property of NMR-active media in that it transforms to its transpose if the static field is reversed.

The equation embodies much of the phenomenology of NMR, as may be seen (*3*) by writing the excitation, in terms of the first order spherical tensors $\xi_\pm = \mathbf{e}_x \pm i\mathbf{e}_y$, as $\mathbf{H}_\pm = H\xi_\pm$. Since we are operating in the laboratory frame of coordinates, a time factor $\exp(-i\omega t)$ is understood, but not written. Then the two spherical tensor forms of **H** are seen to describe quadrature excitation with opposite senses. With these definitions, the matrix product $\chi\mathbf{H}_-$ gives a non-vanishing result, whereas $\chi\mathbf{H}_+$ vanishes. These results are swapped if the tensor is transposed, i.e. if the static field reverses. Also note that Eq, [1] predicts a null response if excitation is applied along the z axis. Then this model describes mathematically the example discussed earlier in which the leads to a quadrature probe are switched.

Our discussion so far has treated signal formation globally, in terms of both transmission and reception. We will now consider reception alone, and give a formula for the signal voltage in NMR, which is essentially a specialized statement of Faraday's law. As such, it is not subject to limits of linear response, and applies for any excitation of the nuclei.

Our derivation does not follow the usual reciprocity arguments, but does make considerable use of the symmetry of Green's functions, which lies at the heart of reciprocity in general. Our starting point is a volume integral -- of vector potential and current density, over the sample volume-- related to the flux φ_s arising from the sample, and the current I_c in the coil:

$$\varphi_s I_c = \int \mathbf{A}_s(r) \bullet \mathbf{J}_c(r) d^3r \quad [2]$$

where the subscripts *s* and *c* specify quantities arising in the sample (i.e. the vector potential due to magnetization) and the coil (i.e. the current density in the windings.) This can be re-written using the quasistatic Green's function $G(r,r') = 1/(4\pi\,|\,r-r'\,|)$, where the primed variable is the source point and unprimed the observation point, as

$$\varphi_s I_c = \mu_0 \int\int \mathbf{J}_s(r') \bullet \mathbf{J}_c(r) G(r,r') d^3r' d^3r = \int \mathbf{A}_c(r') \bullet \mathbf{J}_s(r') d^3r' \quad [3]$$

Note that this equality derives from the symmetry of the Green's function with respect to source and observation. The swapping of subscripts *c* and *s* is typical of reciprocity arguments.

We now write the sample current as the curl of the sample magnetization, to get:

$$\int \mathbf{A}_c(r') \bullet \mathbf{J}_s(r') d^3r' = \int \mathbf{A}_c(r') \bullet \nabla \times \mathbf{M}_s(r') d^3r' \quad [4]$$

which is transformed using the well known vector identity

$$\nabla \bullet (\mathbf{A} \times \mathbf{M}) = \nabla \times \mathbf{M} \bullet \mathbf{A} - \nabla \times \mathbf{A} \bullet \mathbf{M} \qquad [5]$$

Since the left hand side of [5] vanishes when integrated over space (by the divergence theorem) and since the curl of vector potential is the magnetic induction, we easily obtain:

$$\varphi_s I_c = \int \mathbf{A}_c(r') \bullet \mathbf{J}_s(r') d^3r' = \int \mathbf{B}_c(r') \bullet \mathbf{M}_s(r') d^3r' \quad [6]$$

Dividing [6] through by the sample current, and taking the negative time derivative, we obtain the sample emf in terms of the coil B field per unit current, and the sample magnetization, an epochal result first given by Hoult and Richards in their classic paper (*1*) of 1976 .

For a uniform field irradiating a homogeneous sample, the emf resulting from the right hand side of [6] can be written as a bilinear form in terms of the susceptibility tensor and the excitation field $\mathbf{H}$.

$$-\dot{\varphi} = \frac{i\omega\mu_0}{I_c}\mathbf{H}\chi\mathbf{H}V \qquad [7]$$

where V is the sample volume. For irradiation directly on resonance, by a linear field (say of a surface coil) with x and y components this becomes:

$$-\dot{\varphi} = \frac{i\omega\mu_0}{I_c} i\chi''(\omega)(H_x + iH_y)(H_x - iH_y)V \qquad [8]$$

where the complex scalar factors $(H_x + iH_y)$ and $(H_x - iH_y)$ are called the *antenna patterns* for transmission and reception respectively, for this particular coil. Although these are more familiarly known the *sensitivities* (*4*), the term antenna pattern is used here, since these quantities obey a well known principle for radio transmission in gyrotropic media (*2*), namely that the antenna patterns for transmission and reception are exchanged when the static polarizing field reverses direction.

1. D. I. Hoult and R. E. Richards, J. Magn. Reson. **24**, 71 (1976).

2. R. F. Harrington and A.T. Villeneuve, IRE Trans. Microwave Theory and Technique, **MTT6**, 308 (1958).

3. J. Tropp, Phys. Rev. A **74**, 062103 (2006)

4. K. P. Preussman, M. Weiger, M. B. Schiedegger, and P. Boesiger, Magn. Reson. Med. **42**, 952 (1999).