An Information Theoretic Approach to Optimal Q-space Sampling

Hans Knutsson[1] and Carl-Fredrik Westin[2,3]

[1]*Biomedical Engineering, CMIV, Linköping, Östergötland, Sweden,* [2]*Harvard Medical School, MA, United States,* [3]*Linköping University, Östergötland, Sweden*

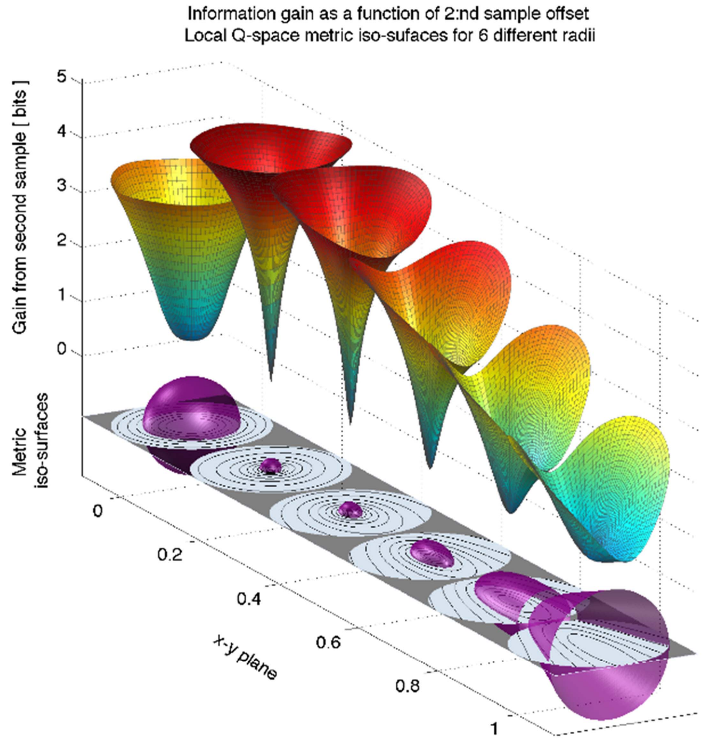
Introduction: The discussion concerning optimal q-space sampling strategies has been lively from the very start of diffusion imaging [1-5]. Here we present a novel approach to determine a local q-space metric that is optimal from an information theoretic perspective with respect to the expected signal statistics. The obtained metric will then serve as a guide for the generation of specific q-space sample distributions e.q. sample distributions obtained in the manner described in [7]. It should be noted that the approach differs significantly from the classical estimation theory approach, e.g. one based on Cramer-Rao bounds [9]. The latter requires a pre defined mathematical representation, the estimator. Our suggestion aims at obtaining the maximum amount of information without enforcing a particular feature representation.

Theory: The mutual information (originally termed 'rate of transmission' [8]) between to signals relates directly to the entropies involved and can be estimated from the joint signal statistics. Using a Gaussian signal+noise source model, which is quite appropriate in the present context, the estimate is directly related to the correlation, ρ_{ab}, between two signals, *a* and *b*, and is given in bits by:

$$I_{ab} = -\tfrac{1}{2}\log_2(1-\rho_{ab}^2)$$

This expression can also be used to estimate the information from a single signal by measuring the correlation between the signal with one noise realization and the same signal with a second uncorrelated noise realization from the same noise source. (The second is denoted by a prime below.) In order to obtain an estimate of a local information based q-space metric we can now compute the information gain, I_Δ, from measuring in a second q-space location, q_b, given that we already have a measure at a first location, q_a. I_Δ is obtained as the information due to the second measurement alone minus the mutual information between the two measured signals, i.e. the information that is already present du to the first measurement:

$$I_\Delta \;=\; I_{bb'} - I_{ab} \;=\; \tfrac{1}{2}\log_2\left(\frac{1-\rho_{ab}^2}{1-\rho_{bb'}^2}\right)$$



Method: To obtain the statistics of the q-space signals we generate a large number of q-space response examples. Using these examples correlation estimates between any two q-space locations, as well as correlations between different instances of the same location, can be estimated. From these correlations the added information from measuring in a second q-space location, given a first measurement in any other location, can be found. The fact each voxel in will contain a huge number of the compartments determining the q-space responses and that a substantial intra voxel variation in compartment size and shape can be expected makes it natural to use a Gaussian as a first approximation of the q-space response magnitude.

The example generator was set to produce 3D Gaussian q-space responses having one long axis and two equal short axes. A distribution of 100 different long axes orientations, evenly distributed to cover all 3D orientations, were generated using the method outlined in [7]. The average size of the compartments was set to vary logarithmically in a range of 1 to 2. The ratio between the long and short axes was set to vary logarithmically in a range of 1 to 10 while keeping the volume constant. The total volume of the examples of a given volume was set to vary as the inverse of the example volume raised to 1.5, i.e. the total volume of the smaller compartments was somewhat greater than the total volume of the larger compartments. The noise level was set to give an average SNR of approximately 5 (14 dB) in the q-space range covered. We believe that this choice of parameters represent a reasonable expectation of in vivo brain data.

Results: The figure shows six results obtained for the initial point located on the x-axis at 6 different q-space radii (0, 0.2, 0.4, 0.6, 0.8 and 1.0). The numbers just indicate a relative distance but are chosen to highlight the typical information gain behaviors that will be present. For each location the multi colored surfaces show the information gain when varying location of the second measurement in the x-y plane. The purple iso-surfaces show the 3D q-space locations where the gained information is 2 bits. Since the setup is rotationally invariant the results will be the same along any axis through the origin. A rough interpretation of the situation at different q-space radii, *r*, is: @ *r* = 0: The second sample must be moved quite far from the first to gain more information, i.e. very sparse sampling is needed, a sample at q=0 picks up most of the information available at the center. @ *r* = 0.2: The information gain now quickly increases in an isotropic fashion when the second location is moved away from the first. This indicates that a relatively dense sampling is preferable here. @ *r* = 0.4: The situation resembles the previous one but a slight anisotropy of the 2-bit iso-surface can be noted. @ *r* = 0.6: The anisotropy is becoming more pronounced indicating that moving the second location in the angular direction is preferable to a change in radius. The volume enclosed by the iso-surface is also larger indicating that a less dense sampling is needed. @ *r* = 0.8: The anisotropy in favor of moving in the angular direction is now quite pronounced indicating that a high number of orientations is needed at this radius. The density in the radial direction, however can be relatively low. @ *r* = 1.0: The iso-surface now takes the shape of a cone implying that the information gain is largest when moving simultaneously inwards and angularly. This is due to that the SNR at this radius quickly decreases with increasing radius due to loss of signal strength.

Conclusions: Although the interpretation of the results may be accordance with the 'gut feeling' of some experienced researchers in the field we believe that our analysis provides a novel view allowing a quantification of said feeling. The presented analysis can, for example, be used to derive parameters for the 3D q-space sample distribution scheme described in [7]. In this way full q-space sampling, optimal with respect to a given expected distribution of diffusion propagators can be produced. This will also allow tuning of q-space distributions to maximize resolution for targeted tissue features.

Grant support: NIH grants P41EB015902, R01MH074794, CADICS Linneaus research environment and Swedish Research Council grant 2012-3682.

References: [1] Wiegell et al., Radiology 217, 2000. [2] Assaf et al., MRM 52, 2004, [3] Wu et al., Neuroimage 36, 2007. [4] Alexander, MRM 60, 2008. [5] Westin et al., ISMRM, 2012. [6] Jones et al., MRM 42, 1999. [7] Knutsson et al., MICCAI, 2013. [8] Shannon, Bell systems, 1948. [9] Rao, Calcutta math. soc., 1945.