**Analogy of phase-constrained parallel MRI formulations**

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**Target audience:** Researchers who are interested in parallel imaging methods and phase constrained algorithms

**Purpose:** One major drawback of parallel MRI (pMRI) is the noise amplification due to the reconstruction process. The noise amplification originates from the formulation of pMRI as an inverse problem and is characterized by the g-factor (1). Phase-constrained algorithms achieve a significant g-factor reduction by constraining all elements of the solution to be real-valued and are categorized into two formulations. In the original formulation, the SENSE reconstruction problem is split into real- and imaginary parts and is referred to as phase-constrained (PC) SENSE (2,3,4). An alternative formulation utilizes conjugate k-space symmetry by generating virtual coils (VC) (5). The purpose of this work is to demonstrate by mathematical analysis that both formulations are equivalent.

**Methods:** An extended encoding matrix $\mathbf{B} = \mathbf{C}\cdot\mathbf{P}$ is defined by combining the spatial coil sensitivity estimates $\mathbf{C}$ with the spatial phase information. The diagonal matrix $\mathbf{P}$ contains the phase estimates at the source locations. Neglecting noise correlations between the receiver channels (i.e. by performing a noise pre-whitening procedure prior to the reconstruction), the PC-SENSE problem and the corresponding g-factor can be written as (3,4):

$$\begin{pmatrix}\mathrm{Re}(\mathbf{s})\\ \mathrm{Im}(\mathbf{s})\end{pmatrix} = \begin{pmatrix}\mathrm{Re}(\mathbf{B})\\ \mathrm{Im}(\mathbf{B})\end{pmatrix}\cdot\begin{pmatrix}\mathrm{Re}(\boldsymbol{\rho})\\ 0\end{pmatrix} \quad \text{and} \quad g_j^{PC} = \sqrt{\left(\left(\mathbf{D}^H\mathbf{D}\right)^{-1}\right)_{jj}\left(\mathbf{C}^H\mathbf{C}\right)_{jj}} \quad \text{with } \mathbf{D} = \begin{pmatrix}\mathrm{Re}(\mathbf{B})\\ \mathrm{Im}(\mathbf{B})\end{pmatrix} \qquad [1]$$

The vector $\mathbf{s}$ represents the measured pixel intensities for all channels and $\rho$ is the solution vector. In contrast, the VC concept works by generating additional synthetic channels that contain the complex conjugate (denoted by *) pixel intensities of the actual channels. The VC-SENSE problem and the corresponding g-factor are written as (5):

$$\begin{pmatrix}\mathbf{s}\\ \mathbf{s}^*\end{pmatrix} = \begin{pmatrix}\mathbf{B}\\ \mathbf{B}^*\end{pmatrix}\cdot\boldsymbol{\rho} \quad \text{and} \quad g_j^{VC} = \sqrt{\left(\left(\mathbf{E}^H\mathbf{E}\right)^{-1}\right)_{jj}\left(\mathbf{E}^H\mathbf{E}\right)_{jj}} \quad \text{with } \mathbf{E} = \begin{pmatrix}\mathbf{B}\\ \mathbf{B}^*\end{pmatrix} \qquad [2]$$

Analogy of both formulations can be demonstrated by comparing the diagonal matrix elements that contribute to the g-factors. It can be shown by mathematical analysis that:

$$\left(\left(\mathbf{E}^H\mathbf{E}\right)^{-1}\right)_{jj} = \frac{1}{2}\left(\left(\mathbf{D}^H\mathbf{D}\right)^{-1}\right)_{jj} \quad \text{and} \quad \left(\mathbf{E}^H\mathbf{E}\right)_{jj} = 2\left(\mathbf{C}^H\mathbf{C}\right)_{jj} \qquad [3]$$

**Results:** The factors ½ and 2 in Eq. [3] are due to the complex-valued variables of twice the number of channels in VC-SENSE whereas PC-SENSE applies real-valued variables. Inserting Eqs [3] into the g-factor in Eq [2] yields the identity of both g-factors. (i.e. $g^{VC} = g^{PC}$). The images shown in Fig. 1 verify the analogy. The PC-SENSE and VC-SENSE image reconstructions as well as the g-factors yield relative differences smaller than $10^{-6}$. These differences can be attributed to numerical inaccuracies. Compared to conventional SENSE, the g-factor is significantly reduced in this example.

**Discussion and conclusions:** Despite their different formulations, PC-SENSE and VC-SENSE lead to identical reconstruction results. Because GRAPPA has been presented using the virtual-coil concept, it can hence be referred to as phase-constrained GRAPPA (6).

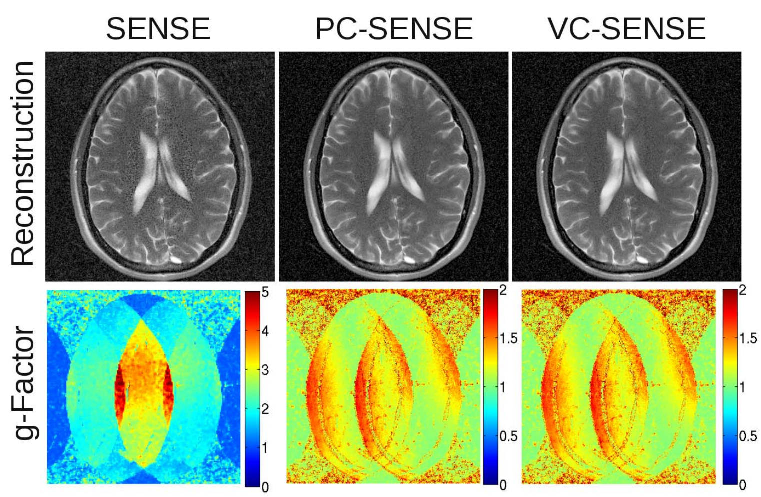


***Figure 1:*** *Image reconstructions (top) and g-factors (bottom) of a 4-fold accelerated scan using conventional SENSE (left), PC-SENSE (middle) and VC-SENSE (right).*

**References:**

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