Bayesian Estimation of Signal Amplitude from Magnitude Data

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Target Audience: MR researchers estimating parameters from low signal-to-noise-ratio (SNR) data.

Purpose: To estimate the underlying signal amplitude from magnitude reconstructed data. Signal intensity after magnitude reconstruction follows a Rician distribution, an estimator of measured signal magnitude given an underlying true amplitude. However, one typically wants the reverse: an estimate of the true amplitude given a measurement of signal magnitude. We outline a derivation of the latter, dubbed the inverse Rician distribution, and examine its properties.

Theory: We *model* the signal as $s_R \equiv A\cos\theta + \eta_R$, $s_I \equiv A\sin\theta + \eta_I$, where s_R, s_I are measured signal intensities in the real and imaginary channels, A is the true signal amplitude, θ is the true phase and η_R, η_I are gaussian noise in each channel with standard deviation σ. We *parameterize* the signal as $s_R \equiv M\cos\varphi$, $s_I \equiv M\sin\varphi$, where M is the measured magnitude and φ is the measured phase. Unlike the model, the parameterization does not distinguish between signal and noise. Using Bayes' rule and assuming flat priors, the joint probability of the true amplitude and phase is:

$$p(A,\theta|M,\sigma,I) \propto \frac{1}{2\pi\sigma^2} exp\left(-\frac{1}{2\sigma^2}\left((Acos\theta - Mcos\varphi)^2 + (Asin\theta - Msin\varphi)^2\right)\right). \quad [1]$$

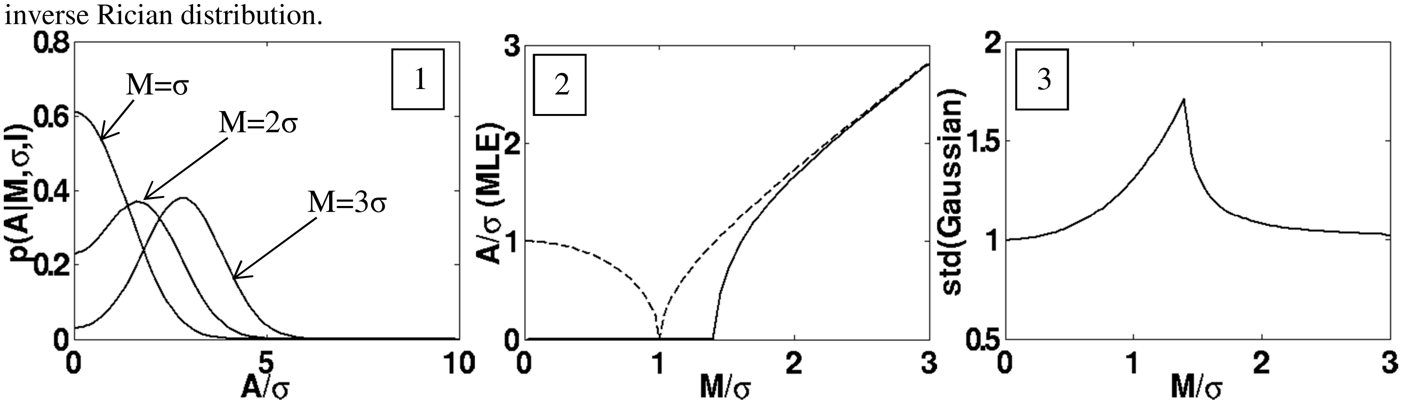
After setting the measured phase $\varphi=0$ and marginalizing over the true phase, θ, the probability distribution for the true amplitude is:

$$p(A|M,\sigma,I) \propto \frac{1}{\sigma^2} exp\left(-\frac{1}{2\sigma^2}(A^2 + M^2)\right) I_0\left(\frac{AM}{\sigma}\right), \quad [2]$$

which is similar to the Rician distribution except for a leading multiplicative term of M.

Methods: We demonstrate properties of the derived probability distribution in comparison with the estimator developed by Gudbjartsson and Patz (GP)[1]. All calculations were performed in MATLAB (the Mathworks, Natick).

Results: Fig. 1 shows the inverse Rician distribution given measured signal magnitudes of 1, 2 and 3 times the noise, σ. Fig. 2 shows the maximum likelihood estimate of true amplitude given different values of measured signal amplitude (solid line) along with the comparable estimate from GP (dashed line). Fig. 3 shows the width of a Gaussian fit to the inverse Rician distribution.



Discussion: The inverse Rician distribution provides a direct estimate of underlying signal amplitude given only a magnitude signal and an estimate of the noise. The maximum likelihood estimate of amplitude is zero for magnitudes less than $\sqrt{2}\sigma$ and differs substantially from the values provided by the GP estimate at SNR < 2 (fig 2). The inverse Rician distribution can also be approximated by a Gaussian with position and standard deviation that can be determined by fast numeric methods.

Conclusion: Magnitude reconstruction images lead to correlations between signal and noise, particularly at low SNR. This correlation leads to biased parameter estimates in diffusion MRI measurements. The inverse Rician distribution provides a straightforward and fast way to disentangle signal from noise given only an signal magnitude and estimate for noise, leading the way to reduced bias in quantitative parameter estimates.

References:

1. Gudbjartsson H, Patz S. The Rician distribution of noisy MRI data. Magn Reson Med. 1995;34(6):910-4.