Fast and robust design of time-optimal k-space trajectories

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Target audience: RF engineers and MR physicists.

Introduction: Non-uniform *k*-space trajectories are needed for many applications in MRI such as design of receive and excitation sequences. Numerical approaches have been proposed to solve this problem [1,2]. However, these are computationally expensive because of the need to finely sample the trajectory, yielding one unknown to be solved per sample point. Dale et al. [3] used an analytical framework for simple trajectories, but it is not clear if this method is applicable to 2D or 3D (only 1D results were shown). A commonly used method is the optimal control (OC) approach proposed by Lustig et al. [4] that numerically computes optimal gradients based on an arbitrarily parameterized *k*-space trajectory $\boldsymbol{k}(a)$, $a \in [0,1]$. This approach performs well for trajectories without sharp turns but is affected by oscillations as well as slew rate and gradient violations in more challenging cases (the computation of the curvature is unstable for interpolated curves with sharp turns). In this work, an analytical approach for the design of complex 3D k-space trajectories is proposed. The presented method does not use interpolation and is therefore robust and fast. It exactly satisfies the gradient system constraints and ensures that the trajectory duration is minimal.

Methods: Given a set of control points $\boldsymbol{k}^{(1)},\ldots,\boldsymbol{k}^{(N)}$ and gradient magnitude and slew rate constraints $|G(t)| \le G_{max}$ and $|S(t)| \le S_{max}$, a *k*-space trajectory $\boldsymbol{k}(t)$ is to be found that intersects the control points at ascending time points $t^{(1)} \le t^{(2)} \le \ldots \le t^{(N)}$ and minimizes the total pulse duration $t^{(N)}$. Consecutive control points $\boldsymbol{k}^{(q)}$ and $\boldsymbol{k}^{(q+1)}$ are connected by piecewise linear gradients (quadratic segments in *k*-space, Fig. 1). The slope of these linear segments directly incorporates the slew rate constraint. Joining neighboring points by these basis functions yields slow overall trajectories because of consecutive decelerations and accelerations (Fig. 1b) even when playing gradients at maximum slew rate. To avoid this situation, the gradient strength at the common point shared by two consecutive segments is optimized (Fig. 1c) so as to minimize the duration of the two segments. For piecewise linear gradients, the duration of a *k*-space move connecting two points can be written explicitly as a function of the 3D gradients at both points. The objective function to be minimized is the sum of the duration of the segments joining consecutive points, which is therefore an explicit function of the gradients at all control points. This objective is minimized using a constrained interior point algorithm under the constraint that the gradient strength is always below G_{max}. The Jacobian of the objective is computed analytically, which greatly speeds up the optimization and makes it less subject to numerical errors. Note that when two control points are very close to one another, small changes in the gradient strength at both points can dramatically change the shape of the smooth curve joining them. This can cause the objective function (which is a measure of the duration of that curve, as explained above) to become discontinuous. To solve this problem, a penalty term is added to the objective that exactly cancels these discontinuities where they occur.

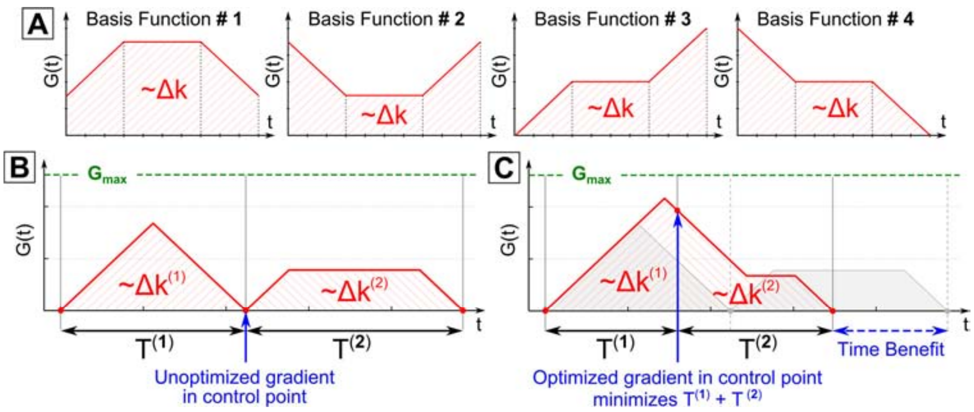

Figure 1: A: Basic gradient shapes used to connect consecutive *k*-space points. **B:** Solution of basis functions such that gradients integrate to $\Delta k^{(1)}/\Delta k^{(2)}$, respectively. **C:** Optimization of the gradient strength at the control point shared by the two segments to globally minimize the pulse duration given by $T^{(1)}+T^{(2)}$

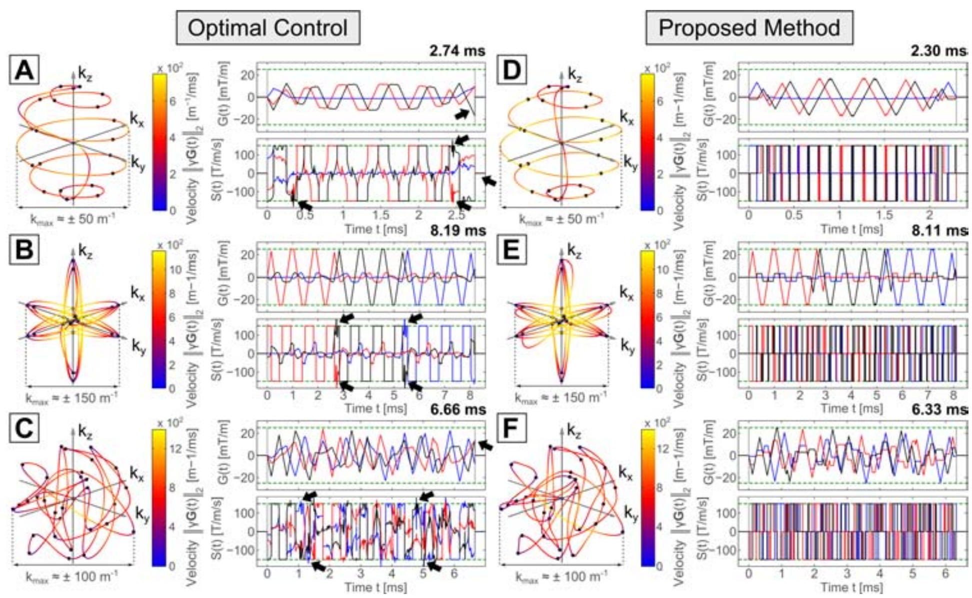

Figure 2: Comparison of 3D *k*-space trajectories designed using the optimal control (OC) method of Lustig et al [4] (spline interpolation) and the proposed method. **A** and **D** show 3D spirals, **B** and **E** show 3D cross shapes with several shells and **C** and **F** show points placed randomly in *k*-space. The gradient constraints were G_{max}=25 mT/m and S_{max}=150 T/m/s. The arrows show constraint violations due to instabilities of the OC approach in the form of oscillations. In contrast, the trajectories computed with the proposed approach satisfy the constraints exactly and are faster.

Results/Discussion: Hundreds of 3D trajectories with different shapes were designed using the proposed method: without exception, the method converged to a high quality solution without oscillations and constraint violations. In contrast, the optimal control (OC) method of Lustig et al. was sometimes unstable and often yielded constraint violations (Fig. 2). Another advantage of the proposed approach is that it was better able to use the gradient system at full performance, therefore yielding shorter pulses than OC (the 3D spiral, 3D cross and random points trajectory were 16%, 1% and 5% faster with the proposed method than with OC, respectively). Additionally, computation time was a few minutes for the OC method, depending on the exact trajectory, compared to a few seconds with the basis function approach.

References: [1] Simonetti et al., IEEE TMI 12(2), 1993, [2] Hargreaves et al., MRM 51(3), 2004, [3] Dale et al., ISMRM, 2002, [4] Lustig et al., IEEE TMI 27(6), 2008