A Fast Algorithm for Rank and Edge Constrained Denoising of Magnitude Diffusion-Weighted Images

Fan Lam[1,2], Ding Liu[1,2], Zhuang Song[3,4], Michael W Weiner[3,4], Norbert Schuff[3,4], and Zhi-Pei Liang[1,2]

[1]*Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, Illinois, United States,* [2]*Beckman Institute, University of Illinois at Urbana-Champaign, Urbana, Illinois, United States,* [3]*Department of Veteran Affairs Medical Center, San Francisco, California, United States,* [4]*Radiology and Biomedical Imaging, University of California, San Francisco, California, United States*

Introduction: Effective denoising of magnitude diffusion imaging data is highly desirable in practice due to a number of practical reasons (such as availability and easier application). One key challenge in denoising magnitude diffusion-weighted (DW) images is to handle the noncentral χ noise model[1,2]. Various methods have been proposed to address this problem from a statistical estimation perspective and achieved impressive denoising results[3-8]. However, these methods are usually computationally demanding because of the need to solve the associated nonlinear optimization problems, limiting their practical utility, especially for processing a large number of 3D high-resolution DW images (DWI). Recently, an efficient quadratic majorize-minimize (MM) scheme was proposed for statistical estimation problems with noncentral χ distributions[9]. In this work, we extend this scheme to non-central χ denoising with joint rank and edge constraints[8]. We show that the resulting new algorithm can achieve similar or slightly better denoising performance compared to a previously proposed BFGS-based algorithm[8], but with significantly reduced computation time.

Theory: Given a sequence of noisy magnitude DW images $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_Q]$, where each vector $\mathbf{y}_q$ stands for one image, the goal in denoising is to estimate the noise-free image sequence $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_Q]$ from $\mathbf{Y}$. We formulate this problem into the following penalized maximum likelihood estimation problem:

$$\hat{\mathbf{U}}, \hat{\mathbf{V}} = \arg\min_{\mathbf{U},\mathbf{V}} L(\mathbf{U},\mathbf{V},\sigma,n) + \lambda R(\mathbf{U},\mathbf{V}) = \arg\min_{\mathbf{U},\mathbf{V}} \sum_{q=1}^{Q}\sum_{m=1}^{M}\left[\frac{[\mathbf{UV}]_{mq}^2}{2\sigma^2} + (n-1)\log\left([\mathbf{UV}]_{mq}\right) - \log I_{n-1}\left(\frac{\mathbf{Y}_{mq}[\mathbf{UV}]_{mq}}{\sigma^2}\right)\right] + \lambda R(\mathbf{U},\mathbf{V}), \quad (1)$$

where $L(.)$ comes from the noncentral χ likelihood model, $\mathbf{U}$ and $\mathbf{V}$ are rank r matrices ($r<Q<<M$), n is the number of coils used for acquisition, $I_n(.)$ is the nth-order modified Bessel function of the first kind, σ^2 is the noise variance, M is the number of voxels in each DW image, Q is the number of images, and $R(.)$ is a regularization function enforcing a joint edge constraint[8,10]. The final denoised images are computed as $\hat{\mathbf{X}} = \hat{\mathbf{U}}\hat{\mathbf{V}}$. Solving the problem in Eq. (1) requires extensive computations. We developed an MM-based fast algorithm to address this issue. Specifically, based on the MM framework[9,11], we derived an upper bound on $L(.)$ to transfer the original optimization problem into a series of simpler problems where the upper bound is minimized. It can be shown that $L(.)$ is upper bounded by

$$L(\mathbf{U},\mathbf{V},n,\sigma^2) \le \sum_{m,q}\left[\frac{[\mathbf{UV}]_{mq}^2}{2\sigma^2} + \left(\frac{n-1}{\mathbf{X}_{mq}^{(i)}} - \frac{I_{n-2}\left(\frac{\mathbf{X}_{mq}^{(i)}\mathbf{Y}_{mq}}{\sigma^2}\right) + I_n\left(\frac{\mathbf{X}_{mq}^{(i)}\mathbf{Y}_{mq}}{\sigma^2}\right)}{I_{n-1}\left(\frac{\mathbf{X}_{mq}^{(i)}\mathbf{Y}_{mq}}{\sigma^2}\right)} \bullet \frac{\mathbf{Y}_{mq}}{2\sigma^2}\right)[\mathbf{UV}]_{mq}\right] + C, \quad (2)$$

where $\mathbf{X}_{mq}^{(i)}$ correspond to the estimated image intensities at the ith iteration and C is a constant. According to Eq. (2), at each iteration, the original problem can be transferred into the following problem $\hat{\mathbf{U}}, \hat{\mathbf{V}} = \mathrm{argmin}_{\mathbf{U},\mathbf{V}} \|\mathbf{UV} - \hat{\mathbf{Y}}\|_F^2 + \lambda R(\mathbf{U},\mathbf{V})$, (3), which is a rank and edge constrained Gaussian denoising problem with modified data $\hat{\mathbf{Y}}$ and can be solved very efficiently using an alternating minimization scheme[12].

Methods: We have evaluated the performance of the proposed algorithm with comparison to a previously proposed BFGS-based algorithm[8] using different sets of data. One of them was from a high quality exvivo pig brain data set[13]. The original DWI series has 64 3D volumes, including 61 diffusion directions at b = 4009mm^2/s and three volumes at b = 0. Each 3D volume has 128x128x70 voxels. Noncentral χ noisy data (with n=1) was simulated from this high SNR data. The joint rank and edge constrained denoising as in Eq. (1) was applied to process the **entire 4D data set simultaneously** with r=12, solved by the BFGS-based algorithm and the proposed MM-based algorithm, respectively. The regularization parameter λ was manually optimized for each algorithm by comparing the results to the high SNR data.

Results: Figure 1 illustrates the denoising performance of the proposed algorithm by comparing the DWIs and tensor estimations from different methods, treating the original high SNR data as a gold standard. As can be seen, the rank and edge constrained formalism produces excellent denoising results, both from the BFGS-based algorithm and the MM-based algorithm. The new algorithm obtains slightly better results with significantly less time (**BFGS: 46491s; MM: 2760s**).

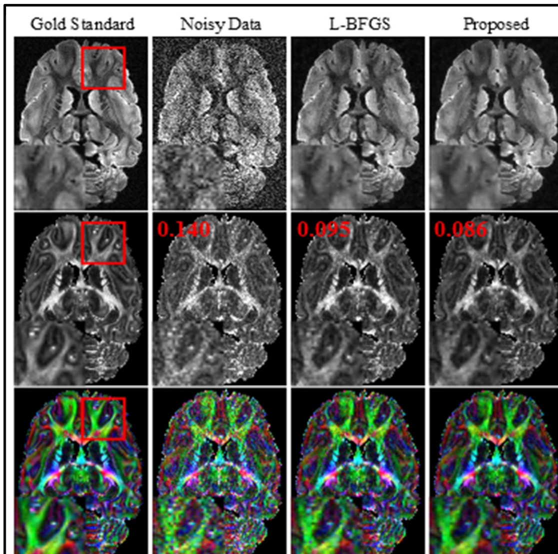

Fig. 1. Comparison of DWIs corresponding to a slice from one diffusion direction (top row), FA maps (middle row) and FA maps color-coded by the direction of the first eigenvector (bottom row), from the gold-standard (column 1), the noisy data (column 2), the BFGS-based algorithm (column 3) and the proposed algorithm (column 4). Zoomed-in regions (highlighted by the red rectangular) are shown in the bottom left corners of each image. The relative root-mean-squared errors for FA values are also shown in red letters.

Conclusion: We have presented a new MM-based algorithm for solving the optimization problem associated with magnitude image denoising with joint rank and edge constraints. The proposed algorithm achieves similar or even better performance compared to a previously used BFGS-based algorithm, but with significantly reduced computation time. We expect the new algorithm to enhance the practical utility of rank and edge constrained denoising and allow for the incorporation of more prior information for further improvements in denoising performance.

Reference: [1] Gudbjartsson et. al., MRM, 1995. [2] Constantinides et. al., MRM, 1997. [3] S Basu et. al., MICCAI, 2006. [4] N Wiest-Daesslé et. al., MICCAI, 2008. [5] Aja-Fernandez et. al., IEEE-TMI, 2008. [6] Descoteaux et. al., MRM, 2006. [7] Manjon et. al., PLoS ONE, 2013. [8] Lam et. al., MRM, 2013. [9] Varadarajan and Haldar, IEEE-ISBI, 2013. [10] Haldar et. al., MRM 2013. [11] Hunter and Lange, Am Stat, 2004. [12] Lam et. al., ISMRM, 2013. [13] Dyrby et. al., HBM, 2011.

Acknowledgement This work was supported in part by the following research grants: NIH-P41-EB015904, NIH-P41-EB001977, and NIH-1RO1-EB013695.