# Gridding & the NUFFT for Non-Cartesian Image Reconstruction

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**Introduction** Typically MRI data is collected on in a rectilinear, or Cartesian, sampling pattern. Image reconstruction can then be performed with a simple 2D (or 3D) discrete Fourier transform. However, there is a long history of acquisition methods using non-Cartesian sampling patterns, going back to the very beginning of MRI. These include spiral and radial acquisition methods as shown in Fig. 1. There are many others, as well as 3D extensions. These require a more sophisticated reconstruction methods, and have been a major area of research in MRI. In this presentation the basic ideas and technical issues involved with these reconstruction methods will be described.

**Problem Statement** There are several related problems here. The fundamental problem is that we have non-uniformly spaced samples in the spatial frequency domain, and we want to generate uniformly space samples in the image domain. An initial interpolation in the frequency domain is followed by a Fourier transform. This presents both problems and opportunities. Interpolation methods that would be suitable in the image domain produce significant artifacts after the transform. However, some of these are easily corrected, and this allows very simple interpolators to be used provided they are designed with the subsequent transform in mind.

The basic problem is illustrated in Fig. 2 for two spokes of a radial acquisition. We would like to determine the value of the Cartesian samples from the adjacent samples from the radial acquisition. There are several approaches we could use. Here we will focus on a class of solutions that approach the problem convolving the acquired data with a kernel, and then resample the data onto the Cartesian grid. This is illustrated in Fig. 3. After the data has been resampled, a 2D DFT takes the data to the image domain. These approaches are generally known as "gridding" reconstructions, and are a special case of the non-uniform FFT (NUFFT). The NUFFT includes many different variations, depending on whether the source domain, destination domain, or both, are non-uniform, and how the kernel is designed.

**Gridding Kernels** The initial paper that started the interest in gridding was by O'Sullivan [1], who was motivated by sinc interpolation. The issues with this are shown in Fig. 4. The convolution with the windowed sinc corresponds to apodization in the image domain, as well as aliasing from the
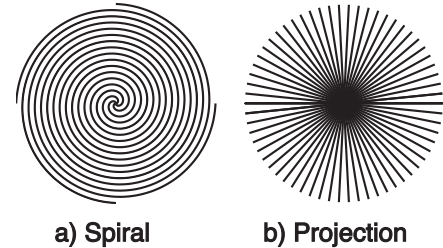


**Fig. 1** Spiral and projection (radial) are the most widely used non-Cartesian sampling patterns, although there are many others.
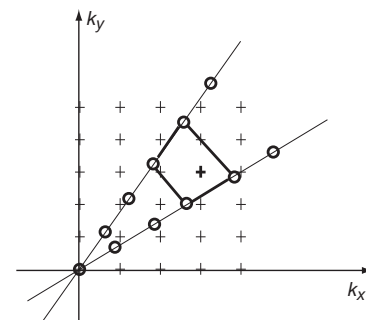


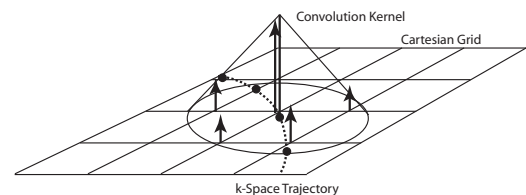**Fig. 2** Interpolating Cartesian data points from samples of a radial acquisition.



**Fig. 3** Gridding kernel convolution and resampling
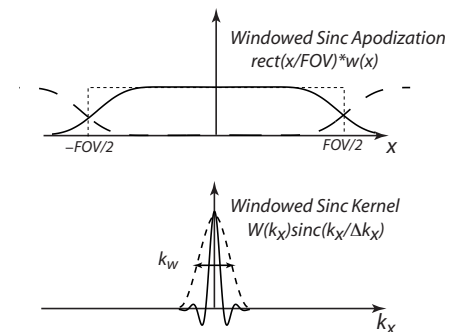


**Fig. 4** A windowed sinc interpolation in spatial frequency results in aliasing in the image domain.

adjacent sampling replicas. This can be minimized by using a larger kernel to contain these effects to the edge of the FOV, but this is expensive in computation time. At the time of ref. [1], this was a major concern. The solution was to use a very simple single lobed gridding kernel. The problem this presents is shown in Fig. 5. The amplitude of the apodization is equal to the amplitude of the aliased signal from the next sampling replica at the FOV/2. The solution O'Sullivan proposed was to sample more finely in spatial frequency to move the sampling replicas out, and provide room for a transition band. This is shown in Fig. 6. Originally this oversampling factor chosen as 2, and this works remarkably well for almost any reasonable kernel. Aliasing is small, and the apodization can be corrected by dividing by the transform of the gridding kernel. In Jackson [2] several kernels were studied, and an optimized Kaiser-Bessel kernel was described. This has been widely used ever since. Several other kernels have been described, including a Gaussian [3] (from the NUFFT literature) and optimized kernels [4,5]. One of the major limitations of this approach is the need to double the number of samples computed in each dimension. For 3D data sets, this is a factor of eight, which can lead to memory issues. A solution is to use a smaller oversampling factor along with an optimized kernel [6]. This provides a continuous tradeoff between computation and memory constraints, and allows very high fidelity reconstructions with oversampling factors as small as 1.25.

**Density Correction** The other issue is the fact that the density of the samples generally varies with non-Cartesian acquisitions. Some spatial frequencies are overrepresented in the data, and if not corrected, this will produce artifacts. The best known example is the rho filter from projection reconstruction. Some correction is required for almost any acquisition method other than conventional spin warp. There are many different approaches. For projection reconstruction there are analytical expressions. For others, the density can be estimated based on an analytical or numerical model. One effective method is the Voronoi diagram, which is part of Matlab. This assigns an area to each sample, that can be used as the density correction factor. An example for a spiral trajectory is shown in Fig. 7.

**Examples** A sequence of reconstructions is shown in Figs. 8-11 for a simple spiral acquisition. With a simple 1X FOV with and without density correction is shown in Fig. 8. Without density correction, low frequency artifacts dominate. With density correction, aliasing artifacts remain. Increasing the oversampling factor by 2 corresponds to doubling the reconstruction FOV. A 2X reconstruction and the central FOV are shown in Fig. 9. This provides a much cleaner reconstruction. Correcting for the apodization gives a much more uniform reconstruction, shown in Fig. 10. The same image quality can be obtained with much less memory, but
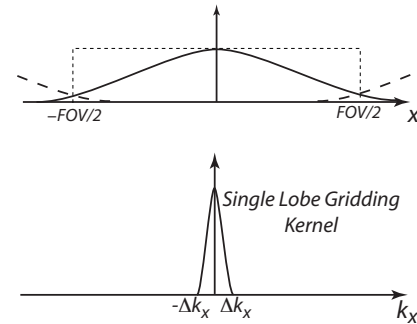


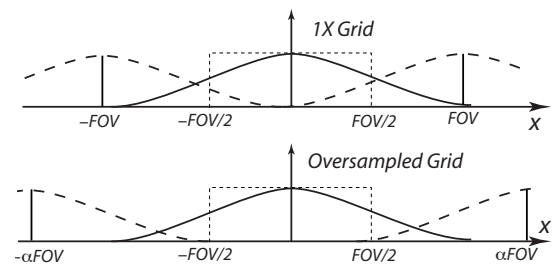**Fig. 5** For a single lobed gridding kernel apodization and aliasing can be traded off, but not eliminated.



**Fig.6** Reconstructing on a more finely sampled Cartesian grid moves the sampling replicas further out, reducing aliasing.
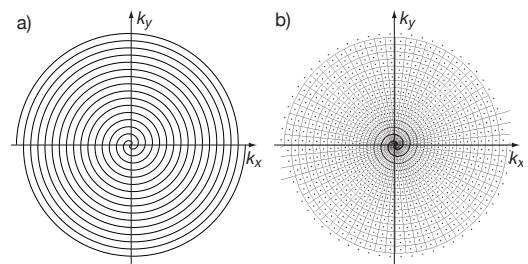


**Fig. 7** Spiral trajectory and its Voronoi diagram. This assigns an area to each sample, which is an estimate of density correction factor.
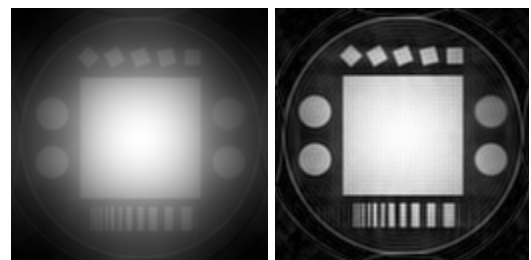


**Fig 8** A simple 1X gridding reconstruction without (left) and with (right) density correction. Significant aliasing artifacts remain.

more computation, using a 1.25X oversampling factor. This is shown in Fig. 11.

**Extensions** So far we have only been concerned with the image reconstruction problem, where we have the non-uniformly spaced k-space data and we want the reconstructed image. Another important NUFFT problem is the inverse of this, where we have the Cartesian sampled image data, and we want to calculate the non-uniform k-space data. This is often called inverse gridding, and is important for iterative reconstruction such as non-Cartesian SENSE [7] or SPIRiT [8]. Inverse gridding proceeds in the reverse order as gridding. The image data is first pre-emphasized to correct for the later convolution in spatial frequency. The data is then transformed to spatial frequency, convolved with the gridding kernel, and resampled on the non-uniform sample points. Because the initial image data is uniform, no density correction is required. This is a significant simplification.

All of the examples we have considered are for 2D images, but the same ideas work for 3D spatial data, and for 4D spatial and temporal data. Typically the gridding kernels are designed to be separable (the product of 1D kernels) to make the analysis easier, but it is also possible to design kernels explicitly for higher dimensions.



**Fig 9** A 2X gridding reconstruction showing the full 2X FOV (left) and the centeral FOV (right). The apodization has not been corrected.
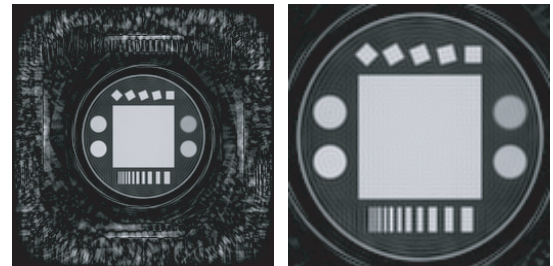


**Fig 10** A 2X gridding reconstruction showing the full 2X FOV (left) and the central FOV (right).
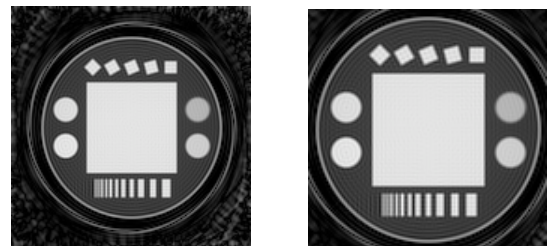


**Fig 11** A 1.25X gridding reconstruction showing the full 1.25X FOV (left) and the central FOV (right). This shows a similar image quality to the 2X FOV reconstruction while requiring much less memory.

**References**
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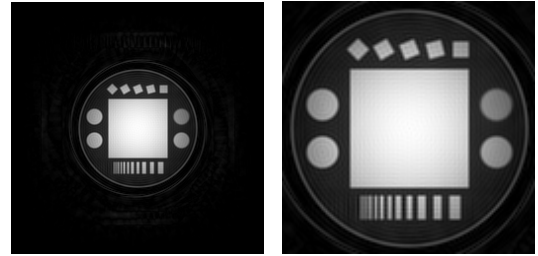8. M.Lustig and J.M.Pauly. "SPIRiT: Iterative self-consistent parallel imaging reconstruction from arbitrary k-space." *Magn Reson Med.* 2010 Aug;64(2):457-71.