Generalized Local Maxwell Tomography for Mapping of Electrical Property Gradients and Tensors

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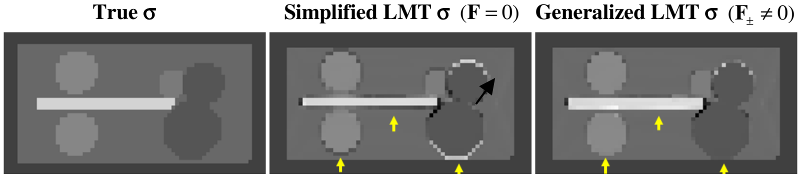
Introduction: At last year's ISMRM meeting, we introduced the Local Maxwell Tomography (LMT) method for noninvasive mapping of the electrical properties of tissue or materials[1]. LMT uses measurements of the curvature of transmit and receive RF fields in MR coil arrays to solve simultaneously for key functions of the missing absolute RF phase distribution and magnetization density distribution along with unknown electrical properties. Unlike prior field-based property mapping methods such as electrical properties tomography (EPT)[2], which may be derived as a special case, the LMT approach is free of assumptions regarding RF phase and coil/field/magnetization structure. However, two simplifying assumptions were retained in our original theoretical description of LMT[1]: a) an assumption of scalar electrical properties, ignoring structural anisotropies which may be found in complex materials, and b) an assumption of piecewise constant or slowly varying properties, ignoring the effects of electrical property gradients upon local electrodynamics. Here, we generalize the theory of LMT to remove these two remaining assumptions. We demonstrate that the generalized framework eliminates edge artifacts observed in simpler implementations, and discuss its potential to enable electrical property tractography, if a sufficient number of measurements and coil elements are deployed.

Methods: *Electrodynamics:* Defining time-harmonic electromagnetic fields $\mathbf{B}(\mathbf{r},t)=\mathbf{B}(\mathbf{r})\exp(-i\omega t)$ and $\mathbf{E}(\mathbf{r},t)=\mathbf{E}(\mathbf{r})\exp(-i\omega t)$ with angular frequency ω, applying the Ohmic relation $\mathbf{J}=\bar{\sigma}\mathbf{E}$ and the constitutive relation $\mathbf{D}=\bar{\varepsilon}\mathbf{E}$ for linear materials to the differential form of Ampère's Law, taking the curl, and substituting Faraday's Law yields the following general equation relating electrical properties to the curvature of magnetic fields: $\nabla\times\left[\left(\bar{\sigma}-i\omega\bar{\varepsilon}\right)^{-1}\left(\nabla\times\bar{\mu}^{-1}\mathbf{B}\right)\right]-i\omega\mathbf{B}=0$. Here, $\bar{\sigma}$, $\bar{\varepsilon}$, and $\bar{\mu}$ are 3x3 matrix representations of electrical conductivity, electric permittivity, and magnetic permeability tensors, respectively. The expressions $\left(\bar{\sigma}-i\omega\bar{\varepsilon}\right)^{-1}$ and $\bar{\mu}^{-1}$ indicate matrix inverses. For the special case in which σ, ε, and μ may be represented as scalar functions of position, and in which the fractional variation in μ is substantially smaller than that in σ and ε (which is generally the case for typical biological tissues and a variety of other common materials), this expression reduces to a generalized form of the familiar Helmholtz equation: $\left(\nabla^2+k^2\right)\mathbf{B}+\mathbf{F}\times\nabla\times\mathbf{B}=0$. Here, we have defined $k^2\equiv i\omega\mu\left(\sigma-i\omega\varepsilon\right)$ and $\mathbf{F}\equiv\nabla\ln\left\{\mu\left(\sigma-i\omega\varepsilon\right)\right\}=\nabla\ln k^2$. *Measurables and inverse problem:* As with the simpler formulation described previously[1], a key to solution of the LMT inverse problem is to separate the right- and left-circularly polarized magnetic field components $B_1^{(\pm)}$ associated with each transmit coil l and each receive coil l' into known (blue) and coil-independent unknown (red) quantities, $B_{1,l}^{(+)}=\left(\left|B_{1,l}^{(+)}\right|\exp\left(i\varphi_{\Sigma_l}\right)\right)\left(\exp\left(-i\varphi_0\right)\right)$ and $B_{1,l'}^{(-)}=\left(\left|MG_{l'}B_{1,l'}^{(-)}\right|\exp\left(i\varphi_{\Delta_{l'}}\right)\right)\left(\left|MG_{l'}\right|^{-1}\exp\left(+i\varphi_0\right)\right)$, where φ_0 is the unknown phase distribution associated with a chosen reference receive coil or coil combination, $|M|$ is the unknown distribution of magnetization, and $G_{l'}$ is the spatially invariant overall gain of each receiver channel. The sum and difference phases $\varphi_{\Sigma_l}\equiv\varphi_{B_{1,l}^{(+)}}+\varphi_0$ and $\varphi_{\Delta_{l'}}\equiv\varphi_{B_{1,l'}^{(-)}}-\varphi_0$ are determined experimentally from selected MR signals or quotients thereof. For slowly varying electrical properties $(\mathbf{F}=0)$, uncoupled Helmholtz equations may be written for each of the field components $B_1^{(\pm)}$, which, following application of the product law of differentiation, results in linear matrix equations that may be solved for the unknowns $\left\{\sigma,\varepsilon,\nabla\varphi_0,\nabla^2\varphi_0,\nabla\ln|M|,\nabla^2\ln|M|\right\}$ at each spatial location[1]. In the more general case, non-vanishing electrical property gradients ($\mathbf{F}\neq 0$) and/or anisotropies in property tensors results in mixing of transverse and longitudinal RF field components. Physically, this mixing corresponds to field perturbations which may be linked to eddy currents and charge accumulations in the vicinity of interfaces or anisotropic structures. *Strategies for solution of the inverse problem:* We first consider the case of scalar properties with non-vanishing local gradients. In this case, expanding the last term in the master equation $\left(\nabla^2+k^2\right)\mathbf{B}+\mathbf{F}\times\nabla\times\mathbf{B}=0$, and defining $F_\pm\equiv\left(F_x\pm iF_y\right)/2$, yields $\left(\nabla^2-\mathbf{F}\cdot\nabla+k^2\right)\mathbf{B}+F_-\nabla B_1^{(+)}+F_+\nabla B_1^{(-)}+F_z\nabla B_z=0$. Since transverse field components $B_1^{(\pm)}$ are known up to coil-independent functions, as shown above, the challenge in generalizing LMT lies in eliminating terms involving the unknown coil-dependent longitudinal RF field component B_{1z}. In order to accomplish this, one may divide the master equation by F_z, then either take a curl or apply a z gradient and use Gauss' Law $\nabla\cdot\mathbf{B}=0$ to derive expressions $\left\{\left(\nabla-\nabla\ln F_z\right)\times\left[\left(\nabla^2-\mathbf{F}\cdot\nabla+k^2\right)\mathbf{B}+F_-\nabla B_1^{(+)}+F_+\nabla B_1^{(-)}\right]\right\}_z=0$ and $\left(\nabla_z-\nabla_z\ln F_z\right)\left[\left(\nabla^2-\mathbf{F}\cdot\nabla+k^2\right)B_\pm+F_-\nabla_\pm B_1^{(+)}+F_+\nabla_\pm B_1^{(-)}\right]-F_z\nabla_\pm\left(\nabla_-B_1^{(+)}+\nabla_+B_1^{(-)}\right)=0$ involving only transverse components. (We have defined $\nabla_z\equiv\partial/\partial z$, $\nabla_\pm\equiv\partial/\partial x\pm i\,\partial/\partial y$.) For each of these relations, we may group terms associated with $B_1^{(+)}$ and $B_1^{(-)}$, respectively, in the general format $O_-B_1^{(+)}+O_+B_1^{(-)}=0$, where $O_\pm$ operators contain derivatives of various orders and coil-independent coefficients involving field-related quantities and electrical properties. To eliminate references to unknown absolute field amplitudes and phases, we may take ratios of expressions for distinct coils and arrive at the following general master equations: $\left\{B_{1l}^{(+)}/B_{1m}^{(+)}\right\}\left\{O_-B_{1l}^{(+)}/B_{1l}^{(+)}\right\}\left\{O_+B_{1m'}^{(-)}/B_{1m'}^{(-)}\right\}-\left\{B_{1l'}^{(-)}/B_{1m'}^{(-)}\right\}\left\{O_+B_{1l'}^{(-)}/B_{1l'}^{(-)}\right\}\left\{O_-B_{1m}^{(+)}/B_{1m}^{(+)}\right\}=0$. Each of the expressions in brackets containing $O_\pm$ may be written in terms of derivatives of logarithms, the expressions separating $B_1^{(\pm)}$ into known and unknown quantities may be inserted, and a nonlinear optimization algorithm of choice may be used to determine the values of the unknowns. If all the useful component equations are used, there are 46 real unknowns and six real equations per transmit-receive coil pair, thus unique solution requires at least eight transceive coils (note that, unlike in the simple $\mathbf{F}=0$ case, both $B_1^{(+)}$ and $B_1^{(-)}$ are required for each coil). One may also use subsets of the component equations, with fewer unknowns but also fewer equations and a reduced degree of over-determination. With the addition of still larger numbers of unknowns, a similar approach to eliminating unmeasured field components B_{1z} (with appropriate derivatives and applications of $\nabla\cdot\mathbf{B}=0$) may be used to solve the *fully general equation for electrical property tensors*. An entirely local formulation, in which an independent solution is derived for each voxel, requires treatment of each derivative order of unknown quantities such as $\left\{\sigma,\varepsilon,\varphi_0,|M|\right\}$ as an independent unknown. However, more global formulations are also possible, in which property values and other unknowns are determined at multiple connected voxels, thereby defining associated derivatives. Such approaches, which are also capable of incorporating additional anatomical or physical constraints, will have higher degrees of over-determination than fully local formulations, but at the price of substantially larger search spaces. *Practical issues and implementation:* A key requirement for robust property mapping with LMT is the determination of accurate numerical derivatives of measured field-related quantities. In our initial implementations, we have used low-order Savitsky-Golay (SG) derivatives with kernel size of 5 to 7 in all three dimensions around each voxel. One practical challenge for generalized LMT is that whenever the outer edge of an SG kernel crosses over a property boundary, derivative estimates at the center of the kernel may be perturbed, resulting in persistent residual artifacts within a kernel width around each discrete edge. We are currently exploring alternative numerical derivative algorithms capable of more accurately modeling the spatially-varying curvature associated with rapidly varying electrical properties.

True σ | **Simplified LMT σ $(\mathbf{F}=0)$** | **Generalized LMT σ $(\mathbf{F}_\pm\neq 0)$**



Results: Figure 1 compares conductivity maps at 7 Tesla field strength in a simplified cylindrical body model with heart, lung, spinal cord, kidney, and muscle compartments, using a simulated 16-element array geometry with 4x2 grids of rectangular elements placed on the anterior and posterior surfaces of the body. As indicated by yellow arrows, a generalized LMT reconstruction (right) removes many of the artifactual stripes at property boundaries seen in a simplified reconstruction incorrectly assuming that $\mathbf{F}=0$ (center). For ease of implementation, the algorithm employed to generate the illustrative results at the right of the figure continued to assume vanishing longitudinal property gradients, yielding notable residual errors in the (left-right) z direction. Additional results removing this simplification will be presented, together with results in structured phantoms and *in vivo*.

Discussion and Conclusions: The LMT approach, in which local gradients of coil-independent unknowns are derived together with electrical property values, is sufficiently general to accommodate spatial variations and anisotropies in conductivity and permittivity. Tracking of anisotropies may eventually result in tractography akin to DTI. Note that LMT represents one embodiment of a new class of algorithms for inverse problem solution based on incomplete interior measurements. Similar approaches may be applied for a variety of other cases in which material properties are determined by producing perturbations and observing responses dictated by partial differential equations. One example currently under investigation involves noncontact spatial mapping of thermal properties of tissue using MR thermometry[3].

References: [1]Sodickson DK et al, ISMRM 2012, 387. [2]Katscher U et al, *IEEE Trans Med Imaging* 2009;28:1365. [3]Alon L et al, ISMRM 2013, submitted.