Exact analytical results for ADC for oscillating diffusion sensitizing gradients

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Introduction Diffusion MRI study of short-length scales in porous media and biological systems is usually based on measuring apparent diffusion coefficient (ADC) and calculating the surface-to-volume ratio *S/V* of restrictions by means of the short-time expression [1]

$$D(t) = D_0 \cdot \left[1 - c \cdot (t/t_D)^{1/2} + O(t/t_D)\right], \quad t_D^{-1} = D_0 (S/dV)^2 \tag{1}$$

(D_0 is the free diffusion coefficient, *d* is the system's dimensionality); the coefficient *c* depends on the time-course of diffusion sensitizing gradient *G*(*t*) (e.g., [1-4]). Eq. (1), generally speaking, is valid at sufficiently short diffusion time, $t \ll t_D$. On the other hand, reliable MR measurements require a sufficient dynamic range of MR signal, i.e., sufficiently high *b*-values. To achieve high *b* in the short-time regime is a technically challenging problem. A promising way to get into the short-time limit is to apply the high-frequency oscillating gradients (OG), for which diffusion time *t* in Eq. (1) can be substituted by the period of a single oscillation $T = 2\pi/\omega$ [5,6], whereas the *b*-value is proportional to a number of oscillations *N* and, therefore, can be high enough. The coefficient *c* for the OG was numerically considered in [7] for large number of oscillations *N*. Its limiting value at $N \to \infty$ was found in [8] for the *cos*-type OG. However, the validity of the short-time (high-frequency) expansion (1) for the OG has never been analyzed. In this communication, we present exact analytical expressions for the coefficient *c* for the OG with an arbitrary number of oscillations *N* and analyze the validity of the high-frequency expansion.

Results We consider the OG $G(\tau) = G_0 \cdot \cos(\omega\tau - \varphi)$, where G_0 is an amplitude, ω is the frequency of oscillations, and φ is an arbitrary phase ($\varphi = 0$ and $\varphi = \pi/2$ correspond to the *cos*- and *sin*-type OG, respectively). The total pulse sequence consists of *N* full periods of oscillations, $t = N \cdot T = N \cdot 2\pi/\omega$. The corresponding *b*-value is $b_N = N \cdot \pi(\gamma G_0)^2 (1 + 2\sin^2\varphi)/\omega^3$. For the OG, it is convenient to rewrite Eq. (1) in the form

$$D(t = 2\pi N/\omega) = D_0 \cdot \left[1 - c'(\omega t_D)^{-1/2} + O(\omega t_D)\right], \quad c' = c \cdot (2\pi N)^{1/2} \tag{2}$$

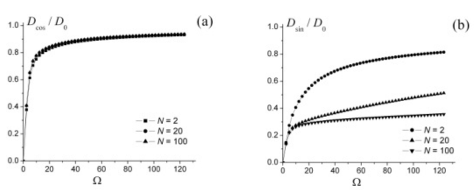
Basing on Eq. (1), the coefficient $c' = c'(\varphi, N)$ is found in the closed analytical form:

$$c'(\varphi, N) = \frac{32\pi N^{3/2}\sin^2\varphi + 12\pi N \cdot C(2N^{1/2}) + 3(3 + 4\sin^2\varphi)\cdot S(2N^{1/2})}{6\sqrt{2}\,\pi N(1 + 2\sin^2\varphi)} \tag{3}$$

where *C*(*x*) and *S*(*x*) are the Fresnel functions. For $\varphi = 0$ the coefficient $c'_{\cos}(N) = c'(0, N)$ varies within a narrow interval: from $c'_{\cos}(N = 1) = 0.81$ to $c'_{\cos}(N \to \infty) = 1/\sqrt{2}$. For any $\varphi \neq 0$, the coefficient $c'(\varphi, N)$ increases at large *N* (as $\sqrt{N}$). The divergence of $c'(\varphi \neq 0, N)$ at large *N* imposes a restriction on the *total* time: $t \ll t_D / \sin^4\varphi$. Thus, for the *cos*-type OG, $c'_{\cos} \sim 1$ for any *N*, that leads to the validity condition $T \ll t_D$, i.e. the period of a *single oscillation* should be smaller than the characteristic diffusion time. Whereas for the *sin*-type OG, Eq. (2) is valid when the *total* diffusion time is smaller than the characteristic time t_D.

It should be reminded, however, that Eq. (3) is derived based on Eq. (1) which is valid only at short *total* diffusion time *t*: $t \ll t_D$, or $N \ll \omega t_D$. However, in some cases, expressions similar to Eq. (2) can be obtained under much "softer" conditions. To demonstrate this, we explore a simple 1D model of restricted geometry, for which an exact solution to the diffusion problem is available for arbitrary time *t*, namely, diffusion within a segment and restricted by impermeable boundaries. For the *cos*-type gradients, ADC is found to be practically independent of *N* (left panel) and is equal to

$$D_{\cos} = D_0 \cdot F(\Omega), \quad F(\Omega) = 1 - \frac{1}{(2\Omega)^{1/2}} \cdot \frac{\sinh\left((2\Omega)^{1/2}\right) + \sin\left((2\Omega)^{1/2}\right)}{\cosh\left((2\Omega)^{1/2}\right) + \cos\left((2\Omega)^{1/2}\right)}, \quad \Omega = \omega t_D \tag{4}$$



The high-frequency behavior of $D_{\cos}$ in Eq. (4) exactly coincides with that obtained from Eqs.(2)-(3). Importantly, Eq. (4) is valid under condition $T \ll t_D$. In contrast, for the *sin*-type OG, $D_{\sin}$ substantially depends on *N* (right panel) and is shown to become independent on *N* and described by the same Eq. (4) only under condition $\Omega \gg N$, i.e. when the *total* time is small enough: $t \ll t_D$. In the intermediate regime, $1 \ll \Omega \ll N$, the function $D_{\sin}/D_0$ is close to 1/3 (see the line corresponding to $N = 100$).

Conclusion The frequency dependence of ADC and the validity condition of the high-frequency expansion (2) are shown to be substantially different for the *cos*- and *sin*-type OG. For the *cos*-type OG, ADC is practically independent of *N* and Eq. (2) is valid when the period of a single oscillation is smaller than the characteristic diffusion time. In contrast, for the *sin*-type OG, ADC substantially depends on a number of oscillations *N*. The high-frequency expansion (2) for the *sin*-type gradients is valid when the *total* diffusion time is smaller than t_D, and the coefficient c' substantially depends on *N*.

[1] P.P. Mitra, et al, Phys Rev B47 (1993) 8565. [2] T.M. de Swiet, P.N. Sen, J Chem Phys 100 (1994) 5597. [3] L.J. Zielinski, P.N. Sen, JMR 165 (2003) 153. [4] S. Axelrod, P.N. Sen, J Chem Phys 114 (2001) 6878. [5] J. Stepišnik, et al, JMR 182 (2006) 195. [6] M.D. Does, et al, MRM 49 (2003) 206. [7]. E.C. Parsons, Jr. et al, MRM 55 (2006) 75. [8] D.S. Novikov, V.G. Kiselev, JMR 210 (2011) 141.