Sparse Tikhonov-Regularized SENSE MRI Reconstruction

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INTRODUCTION: The reconstruction of MRI data acquired using parallel approaches can be considered as solving a linear system formulated through the sensitivity encoding (SENSE) approach in the image domain [1]. Under this approach, the system can become ill-conditioned as the reduction factor increases, and residual aliasing artifacts and noise amplification become more significant. Regularization represents an important technique to overcome this problem. A well-known regularization method is Tikhonov regularization [2]; however, direct Tikhonov regularization is computationally demanding, especially for obtaining the optimal regularization parameter [2]. Such a limitation is expected to grow with SENSE reduction factor. Here, we present a pre-computation-allowable sparse Tikhonov-regularized SENSE MRI reconstruction technique based on QR decomposition, fast regularization parameter estimation using a new L-curve [4], and sparse matrix representation.

METHOD: Our approach to Tikhonov-regularized SENSE reconstruction consists of "pre-computation" and "reconstruction" steps for the full-FOV image, x, from the reduced-FOV image, y. The reduced-FOV image is a folded version of the full-FOV image and formulated as $y = Ax$, where A is a folding matrix. Considering SNR optimization from [1] and [2], this system can be represented as $\tilde{y} = \tilde{A}x$, with whitened observation $\tilde{y} = \Lambda^{-1/2}Q^{H}y$, whitened folding matrix $\tilde{A} = \Lambda^{-1/2}Q^{H}A$, receiver noise covariance matrix $\Psi = Q\Lambda Q^{H}$, and H denotes the transposed complex conjugate. From this, the Tikhonov-regularized solution can be written as $x^{*} = \underset{x}{\operatorname{argmin}}\{\|\tilde{A}x - \tilde{y}\|_2^2 + \lambda^2\|L(x - x_0)\|_2^2\}$, where x_0 is the prior information of x, L is the Tikhonov matrix, and λ^2 is a regularization parameter. The analytical Tikhonov-regularized solution in matrix form is $x^{*} = V\Gamma U^{H}\tilde{y} + V\Phi U^{H}x_0$ (**Eq 1**), where $L = I$, $\tilde{A} = UDV^{H}$, $\Gamma_{ii} = \sigma_i/(\sigma_i^2 + \lambda^2)$, $\Phi_{ii} = \lambda^2/(\sigma_i^2 + \lambda^2)$, and σ_i denotes the ith diagonal elements of D. For the "pre-computation" step, there are three techniques to reduce the computational burden. **1) QR Decomposition**: One major problem of singular value decomposition (SVD) is its computational cost. Therefore, we estimate singular values and vectors by two applications of QR decomposition rather than a single SVD factorization. Thus we consider the decomposition of $\tilde{A}$ as $\tilde{A} = \hat{U}\hat{D}RW^{H} = \hat{U}\hat{D}\hat{V}^{H}$, where $\hat{U}$, $\hat{D}$, and $\hat{V}$ are approximated U, D, and V; and R is a well-conditioned upper triangular matrix. Then **Eq 1** is modified to obtain $x^{*} = \hat{V}\hat{\Gamma}\hat{U}^{H}\tilde{y} + \hat{V}\hat{\Phi}\hat{U}^{H}x_0$ (**Eq 2**), where $L = I$, $\tilde{A} = \hat{U}\hat{D}\hat{V}^{H}$, $\hat{\Gamma}_{ii} = \hat{\sigma}_i/(\hat{\sigma}_i^2 + \lambda^2)$, $\hat{\Phi}_{ii} = \lambda^2/(\hat{\sigma}_i^2 + \lambda^2)$, and $\hat{\sigma}_i$ denotes the ith diagonal elements of $\hat{D}$. This was accomplished by modifying the original framework in [3] to achieve computational cost reduction. **2) Fast Regularization Parameter Estimation Using A New L-curve**: From the traditional L-curve regularization parameter estimation method [5], we estimate the regularization parameter by finding the corner of the L-shape curve from $(\|x_\lambda - x_0\|_2^2, \|\tilde{A}x_\lambda - \tilde{y}\|_2^2)$, where $\|x_\lambda - x_0\|_2^2 = \sum_{i=1}^{n}\hat{f}_i^2\left(\frac{\hat{u}_i^{H}\tilde{y}}{\hat{\sigma}_i^2} - x_0\right)^2$, $\|\tilde{A}x_\lambda - \tilde{y}\|_2^2 = \sum_{i=1}^{n}(1 - \hat{f}_i)^2(\hat{u}_i^{H}\tilde{y})^2$, and $\hat{f}_i = \frac{\hat{\sigma}_i^2}{\hat{\sigma}_i^2 + \lambda^2}$. More efficiently, we use $(\lambda^2, \|x_\lambda - x_0\|_2^2)$ to form the L-curve and calculate its curvature [4]. Note, however, that we cannot precisely compute $\|x_\lambda - x_0\|_2^2$ during the pre-computation step and thus approximate $\hat{u}_i^{H}\tilde{y}$ as $\hat{\sigma}_i^{p+1}$, where p is a behavior-controlling real number [5]. **3) Sparse Matrix Representation**: The SMT (Sparse Matrix Transformation) has shown performance benefits in reconstruction [6]; however, the computational burden to sparsify the matrix during the pre-computation step is high. Here, we propose a Matrix Sparsifier (MS) which is an element-wise weighted sampling method based on the calculated probability of each element. It is computationally efficient and takes advantage of the structure of the regularized inverse of the block-diagonal matrix A. If we pre-compute $H \triangleq \hat{V}\hat{\Gamma}\hat{U}^{H}$ and $h = \triangleq \hat{\Phi}\hat{U}^{H}x_0$ (described in **Eq 2**), the reconstruction step consists of a simple matrix-vector multiplication and vector summation: $x^{*} = H\tilde{y} + h$. To further reduce the reconstruction computational complexity, we sparsify the dense matrix H to generate sparse matrix $\hat{H}$, further simplifying computation of x^{*}.

RESULT: Using our proposed methods on 256x256 simulation data obtained from http://www.nmr.mgh.harvard.edu/~fhlin (3T MPRAGE images obtained with an 8-channel head coil array), the computational cost for the actual reconstruction is reduced by 71% when the reduction factor R=3 and by 98% when R=4 (see **Table 1**), with good image quality and visible reduction in amplified noise and residual aliasing artifacts (see **Fig 1**).

CONCLUSION: We have presented a sparse Tikhonov-regularized SENSE technique that accelerates image reconstruction through pre-computation and sparsification of the dense inverse matrix, and significantly reduces residual aliasing artifacts and noise amplification for ill-posed cases (e.g. when the reduction factor approaches the number of coils).

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	S.T. Reg. (R=3)	S.T. Reg. (R=4)
RMSE	0.016531 (No Reg. = 0.019323)	0.019619 (No Reg. = 0.042852)
Computational Reduction (%)	71.1	98.3

Table 1. Error and computational reduction

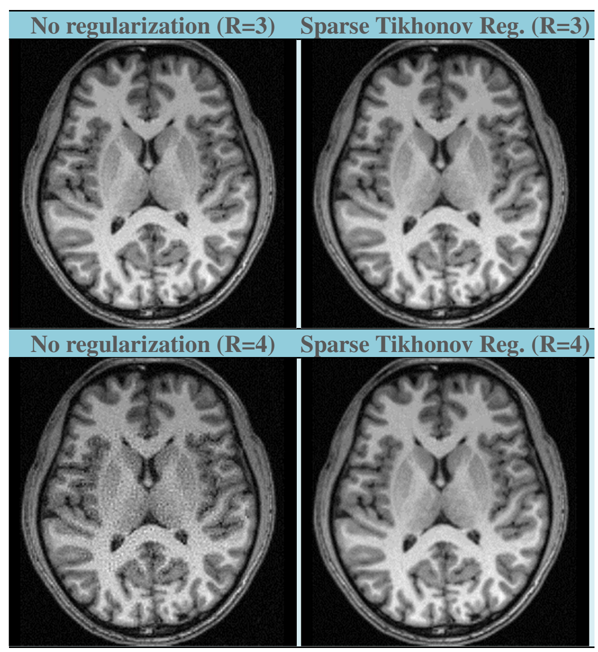

Fig 1. Reconstructed images