Empirical Investigation of the Gardner Transform as a Sparsifying Transform for the Analysis of a New Class of Signals Using Compressed Sensing

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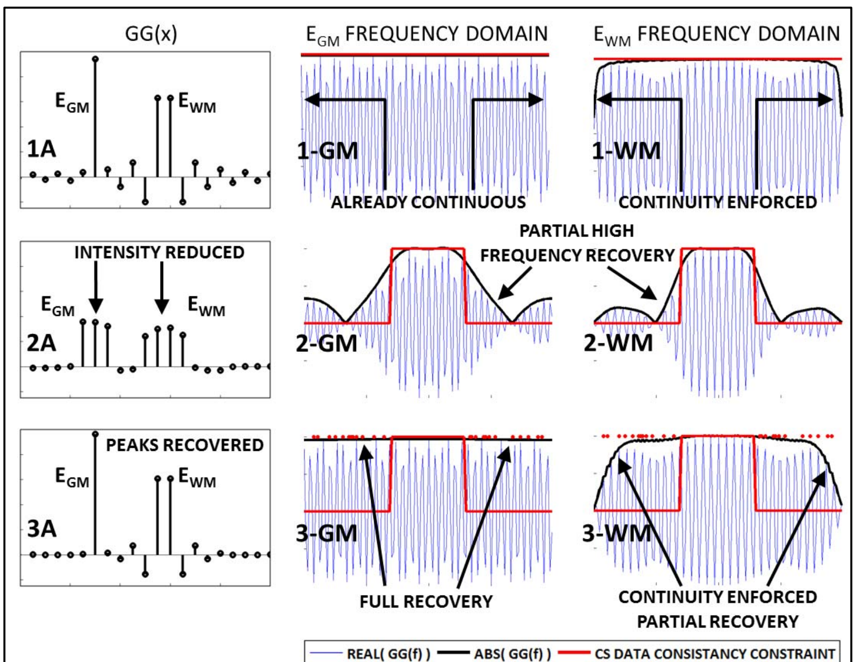
INTRODUCTION: Combining *k*-space under-sampling with a non-linear compressed sensing (CS) algorithm is useful in magnetic resonance (MR) imaging to decrease image acquisition times while retaining resolution[1]. One CS reconstruction criterion is a sparse transform domain representation. The wavelet transform can sparsify images but is not applicable when under-sampling a 4th non-*k*-space domain, time or frequency. In these domains a signal $R(t)$, or its discrete Fourier transform (DFT) $R(f) = DFT(R(t))$, is a sum of exponentials or Lorentzians respectively.

PURPOSE: We propose the use of the Gardner transform[2,3] (GT) as a sparsification transform for sums of exponentials, Lorentzians, Gaussians, or sinc functions. We present GT theory together with simulation results illustrating challenges arising in the practical application of the GT.

THEORY: Analysing a multi-component exponential decay, $R(t) = \sum\{ A_i*exp(-t/MTT_i) \}$ using the GT requires determining the solution of the deconvolution equation, $GG(f) = DFT\{ RR(x) \} / DFT\{ KK(x) \}$, where $RR(x)$ is obtained by the substitution of $t = exp(-x)$ into $RR(t) = t*R(t)$, with the Gardner kernel, $KK(t) = t*exp(-t)$. $GG(x) = IDFT\{ GG(f) \}$ is a sum of delta functions at $x_i = -ln(1 / MTT_i)$ with magnitudes proportional to A_i*MTT_i for the signal $R(t) = \sum\{ A_i*exp(-t/MTT_i) \}$. $RR(t)$ can also be generated from sums of Lorentzians, Gaussian, or sinc functions.

METHODS: Smith et al.[3] showed key GT practical issues were: 1) Gardner domain signal $RR(x)$ generation from either the (exponential) time domain signal, or the (Lorentzian) frequency domain signal, and 2) generating the sparsified signal $GG(x)$ by deconvolving $RR(x)$ by the Gardner kernel $KK(x)$. Stabilizing the deconvolution requires removing high frequency noise components in $GG(f)$. However, this filtering operation also removes high frequency signal components and widens the peaks in $GG(x)$ reducing sparseness. In this investigation we focus on techniques to recover the sparseness lost by filtering by recovering the under-sampled, in this case truncated, $GG(f)$ signal components through the use of a CS reconstruction approach based on *SparseMRI* software[1] . We generated the ideal $GG(f)$ signals for a hypothetical multi-exponential signal obtained from an MR time series containing both gray matter (GM) and white matter (WM) components due to pixel averaging or other partial volume effect. The resulting signal has two exponential components, $R(t) = E_{GM}+E_{WM}$ where $E_{GM} = exp(-t/MTT_{GM})$ and $E_{WM} = exp(-t/MTT_{WM})$; $MTT_{GM} = 4.055s$, $MTT_{WM} = 4.8$s. $GG(f\Delta f)$ was calculated for $-N/2 <= f < N/2$, with $N=128$, $\Delta f=1/(N\Delta x)$, and a GT exponential spectral resolution of $\Delta x = 0.05$. We examined the ideal case of applying CS reconstruction when (1) all $GG(f\Delta f)$ signal components where presumed known for $–N/2 <= f < -N/2$, (2) when $GG(f)$ was low pass filtered, truncated to 30%, matching GT deconvolution, and (3) when CS reconstruction was performed on the truncated signal combined with a CS preconditioning signal; assumed to be derived by modeling of the known central $GG(f)$ core.

RESULTS: Figure 1A is $GT\{R(t) = E_{GM} + E_{WM}\}$ when all $GG(f)$ components are known. The reason behind the peak value and shape differences arise from specific characteristics of the GT frequency components of E_{GM} and E_{WM}, figure 1-GM and 1-WM respectively. $GT\{E_{GM}(f)\}$is a basis function for an N point DFT since $MTT_{GM} = 4.055 = exp(28\Delta x)$. However, $MTT_{WM} = 4.8 = exp(31.4\Delta x)$ so that $GT\{E_{WM}(f)\}$ is not a basis function and has discontinuities at the frequency domain boundaries, $|f| = N/2$. *SparseMRI* CS reconstruction attempts to minimize $| GG(x) |_{L1\text{-norm}}$ widening the E_{WM} peak rather than sharpening it. Note the re-introduced $GG(f)$ continuity[4]. Application of GT-CS on the truncated $GG(f)$ leads to significant intensity and GT resolution loss, figure 2A, and fails to recover missing high frequency components, figures 2-GM and 2-WM. This again can be explained smaller L_1-norms leading to short and wide, rather than tall and narrow, peaks. Preconditioning the missing $GG(f)$ using values obtained by modeling the central $GG(f)$ does not lead to an improved $GG(x)$ estimate unless a few random $GG(f)$ values are assumed known (red-spots in figures 3-GM and 3-WM). Figure 3A is almost exactly recovered to the result in figure 1A despite using half the frequency information. Note the lack of *k*-space continuity[4] in $E_{WM}(f)$ again leads to a wider $GG(x)$ peak in figure 3A.



Column A: GG(x) CS reconstructions for: 1) 100% sampling, 2) center 30% used, 3) center 30% used and 10% random sampling on each side. Column GM and column WM: Frequency magnitude response (black), real component of *GG(f)* (blue), and CS data consistency constraint (red) for E_{GM} and E_{WM} respectively.

CONCLUSION: We have identified the GT as a sparse representation for several non-*k*-space signals, e.g. multi-exponentials, and provided preliminary simulations results. Combining GT-CS with *k*-space extrapolation techniques[5,6] are being investigated.

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