

# Fast Non-Convex Statistical Compressed Sensing MRI Reconstruction Based on Approximated $\ell_p(0 < p < 1)$ -Quasi-Norm With Fewer Measurements Than Using $\ell_1$ -Norm

Il Yong Chun<sup>1</sup> and Thomas Talavage<sup>1,2</sup>

<sup>1</sup>School of Electrical and Computer Engineering, Purdue University, West Lafayette, Indiana, United States, <sup>2</sup>Weldon School of Biomedical Engineering, Purdue University, West Lafayette, Indiana, United States

**INTRODUCTION:** With the help of  $\ell_1$ -norm-based convex optimization and the restricted isometry property (RIP) of a sensing matrix [1], compressed sensing (CS) has become a practical approach to reconstruct sparse images or signals from few samples. CS can be applied to MRI to allow reconstruction from many fewer k-space samples, resulting in a remarkable reduction of MRI scan time [2-4]. However, perfect reconstruction in applications such as MRI is often not achievable, as the RIP is too strong and the solutions of  $\ell_1$ - and  $\ell_0$ -minimization are no longer identical. Instead, the weaker RIP based on  $\ell_p$  ( $p \rightarrow 0$ ) is sufficient to guarantee perfect signal reconstruction [5]. However,  $\ell_p$  minimization is an NP-hard problem because it requires combinatorial optimization. To make this approach tractable, we propose a fast constrained  $\ell_{p,\epsilon}$ - $\ell_2$ -norm ( $\ell_{p,\epsilon}$  is an approximated  $\ell_p$ -quasi-norm) minimization algorithm, based on 1)  $p$ - and  $\epsilon$ -dependent weighting techniques, and 2) an efficient split Bregman-based (SB-based; known to have rapid convergence, especially with an  $\ell_1$ -norm [6]) reweighted  $\ell_1$ -minimization algorithm. This  $\ell_{p,\epsilon}$ - $\ell_2$ -norm minimization achieves exact reconstruction from fewer measurements than are required for the  $\ell_1$ - $\ell_2$ -norm case.

**METHOD:** The partial collection of k-space samples with zero-mean Gaussian noise is formulated as  $y = Ax + n$ , where  $y$  is the observed partial k-space data,  $x$  is the image we want to reconstruct,  $A$  is a partial Fourier transform matrix, and  $n \sim N(0, \Lambda^{-1})$ . Based on the  $\ell_{p,\epsilon}$ -norm, the solution can be written as  $x^* = \underset{x}{\operatorname{argmin}} \|\psi x\|_{p,\epsilon}^p$ , subject to  $\|Ax - y\|_\Lambda^2 < \epsilon$ , where  $\|x\|_{p,\epsilon}^p = \sum_{i=1}^n (|x_i| + \epsilon)^p$ ,  $p \in (0,1)$ ,  $\psi$  is a linear sparse representation transform matrix, and  $\epsilon$  represents the accuracy between the measured data and the reconstruction. We note that  $\epsilon$  is introduced into the  $\ell_p$ -quasi-norm in an effort to escape from local minima and approach the global minimum.

**1) Weighting Technique:** Based on the Majorization-Minimization (MM) algorithm, the norm above is equivalent to  $x^{(k+1)} = \underset{x}{\operatorname{argmin}} \|W^{(k)}\psi x\|_1$ , subject to  $\|Ax - y\|_\Lambda^2 < \epsilon$ , where  $W^{(k)} = \operatorname{diag}(w^{(k)})$ ,  $w_i^{(k)} = p / (|\psi x^{(k)}|_i + \epsilon)^{1-p}$ , and  $i = 1, \dots, n$ .

**2) Reweighted SB:** Using a Bregman iteration and SB technique, this constrained problem can be solved by a series of inner updates and an outer update: **(In-1)**  $x^{(k+1)} = \operatorname{inv}(K)z$ , where  $K \triangleq \mu A^T \Lambda A + \gamma \psi^T \psi$ ,  $z \triangleq \mu A^T \Lambda y + \gamma \psi^T (d^{(k)} - b^{(k)})$ , and  $\operatorname{inv}(\cdot)$  is a technique to invert  $K$  and solve  $Kx^{(k+1)} = z$ ; **(In-2)**  $d_i^{(k+1)} = \operatorname{softshrink}((\psi x)_i^{(k+1)} + b_i^{(k)}, w_i^{(k)} / \mu)$ , where  $\operatorname{softshrink}(x, \alpha) = \frac{x}{|\alpha|} \max(|x| - \alpha, 0)$ ; **(In-3)**  $b^{(k+1)} = b^{(k)} + \psi x^{(k+1)} - d^{(k+1)}$ ; **(Out)**  $y^{(k+1)} = y^{(k)} + y - Ax^{(k+1)}$ . The **(In-2)** can be computed efficiently by the element-wise *softshrink* operator. The total reconstruction time thus depends on the required computations to solve **(In-1)**, which is to be solved analytically.

Note that it is impossible to invert  $K$  directly because of the enormous size. However, if  $\Lambda = \sigma^{-2}I$ , where  $\sigma^2$  is estimated noise variance, and  $\psi$  is the discrete Haar orthogonal wavelet transform ( $\psi^T \psi = I$ ), then  $K$  is a circulant matrix because of the structure of the given  $A$ , and we can solve **(In-1)** efficiently, using only two Fourier transforms. We should note that the weighting matrix  $W^{(k)}$  is updated when each inner loop is completed. Finally, our algorithm will continue until  $\|Ax^{(k)} - y\|_\Lambda^2$  satisfies  $\sigma^2 \epsilon$ ; thus, the stopping criteria is below the expected noise variance, where  $\epsilon \in (0,1)$ .

**RESULT:** We generated Fourier-space samples of a  $256 \times 256$  Shepp-Logan phantom, along 22 radial lines, corresponding to 9% coverage of full k-space. These samples are perturbed by zero-mean Gaussian noise with  $\sigma_n^2 = 10^{-4}$ . Using 10 inner iterations and 140 outer iterations, we performed both  $\ell_1$ - $\ell_2$  convex and the proposed  $\ell_{p,\epsilon}$ - $\ell_2$  non-convex CS-MRI reconstruction algorithms ( $p=0.1$ ,  $\epsilon=0.05$ ) and compared results. With the same number of measurements and iterations, our  $\ell_{p,\epsilon}$ - $\ell_2$  algorithm can accomplish perfect reconstruction while the  $\ell_1$ - $\ell_2$  minimization cannot (**Table 1, Fig 1**). This result is graphically demonstrated as a function of iteration in **Fig 2b**, with the benefit of, introduction

of  $\epsilon$  providing a reasonable ability of escaping from local minima (**Fig 2a**).

**CONCLUSION:** We have presented an efficient non-convex CS-MRI reconstruction algorithm by solving a constrained approximated  $\ell_p$ - $\ell_2$ -norm ( $p \rightarrow 0$ ) minimization problem based on the MM and SB techniques. By introducing an approximation parameter, the proposed algorithm has an ability to approach the global minimum robustly. Using this algorithm we can accomplish rapid, perfect image reconstruction with fewer required measurements than using a  $\ell_1$ - $\ell_2$  CS-MRI reconstruction algorithm, reducing MR imaging time for comparable image quality.

**REFERENCE:** [1] E. Candès, C. R. Acad. Sci. Paris, Ser. I 2008; 346(9-10):589. [2] S. Ramani & J. Fessler, IEEE Trans on Med Imag 2011; 30(3):694. [3] M. Lustig & J. Pauly, Magn Reson Med 2010; 64(2):457. [4] Jung & Hong, et al., Magn Reson Medicine 2008; 61(1):103. [5] R. Chartrand & V. Staneve, Inv Prob 2008; 24(3):14. [6] T. Goldstein & S. Osher, SIAM J. Imaging Sci 2009; 2(2):323.

	$\ell_1$ - $\ell_2$ Conv. CS-MRI	$\ell_{0.1}$ - $\ell_2$ Non-conv. CS-MRI
Iter. (In/Out)	10 / 140	10 / 140
RMSE	0.001712	1.679e-07

Table 1. Error

$\ell_1$ - $\ell_2$ Conv. CS-MRI	$\ell_{0.1}$ - $\ell_2$ Non-conv. CS-MRI
$\mu = 10^5, \gamma = 3 \times 10^2, \sigma^2 = 1$	$\mu = 10^5, \gamma = 3 \times 10^2, \sigma^2 = 1, \epsilon = 0.05$ .

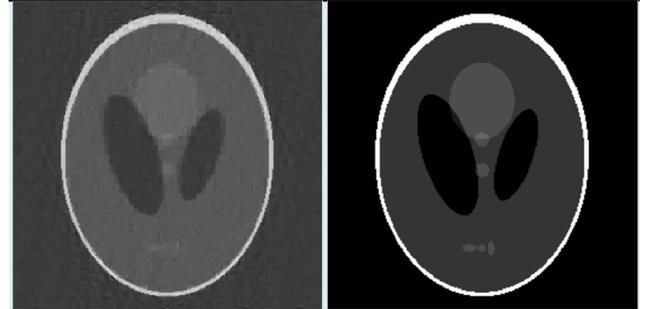


Fig 1. Reconstructed images

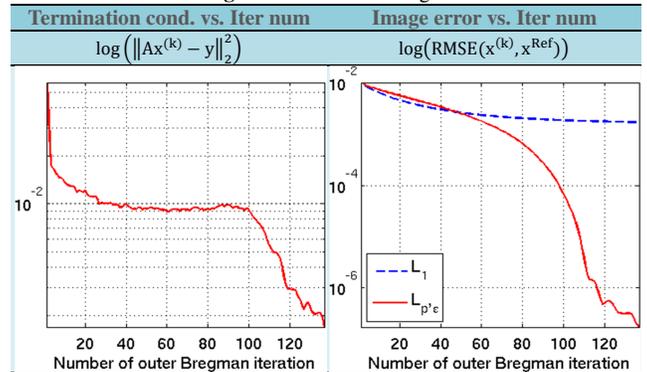


Fig 2. (a) Min behavior

(b) Min behavior comparison