Compressive Diffusion MRI – Part 3: Prior-Image Constrained Low-Rank Model (PCLR)

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Purpose: In another submitted abstract "*Compressive Diffusion MRI – Part 1: Why Low-Rank?*", we compared several sparsity models and found the low-rank (LR) model is the most suitable for diffusion MRI. In this work, assuming that prior images are available, we introduce the Prior-image Constrained LR (PCLR) model, through which prior images can be efficiently utilized to improve LR.

Methods: Let $X_0=[x_1 \dots x_p]$ be the prior images, and $X=[x_1 \dots x_q]$ be the diffusion-weighted (DW) images to be reconstructed. PCLR is formulated as

$$X = \arg\min_X \| PFX - Y \|_2^2 + \lambda \| [X \quad X_0] \|_* , \tag{1}$$

where F denotes the Fourier Transform, P corresponds to the k-space undersampling pattern, and Y is the undersampled k-space data. Here the nuclear norm $\|.\|_*$ regularizes the rank on the augmented matrix $[X\ X_0]$ of X to enforce not only the image similarity among X along the DW dimension, but also the similarity of X with X_0. Note that Eq. (1) is reduced to the standard LR model when X_0 contains no prior image.

Eq. (1) can be transformed into the following convex optimization problem with $X'=[X\ \ X_0]$, $RX'=X$, $R_0X'=X_0$, $Y'=[Y\ \ X_0]$, and $AX'=[PFX\ \ X_0]$

$$X' = \arg\min_{X'} \| AX' - Y' \|_2^2 + \lambda \| X' \|_* \text{ subject to } R_0 X' = X_0 . \tag{2}$$

The convex problem (2) can be solved through the following loop with $f_0'=0$

$$\begin{aligned} X'_{n+1/2} &= \arg\min_{X'} \| AX' - Y' + f'_n \|_2^2 \\ X'_{n+1} &= \arg\min_{X'} \lambda \| X' \|_* + \| X' - X'_{n+1/2} \|_2^2 \text{ subject to } R_0 X'_{n+1} = X_0 \\ f'_{n+1} &= f'_n + AX'_{n+1} - Y' \end{aligned} \tag{3}$$

Here the 1st equation in (3) is equivalent to the update

$$X'_{n+1/2} = [X_{n+1/2} \quad X_0], \text{ with } X_{n+1/2} = \arg\min_X \| FX - (I-P)FX_n - Y + f_n \|_2^2 . \tag{4}$$

And thus we have the following simple-to-implement loop for solving Eq. (1)

$$\begin{aligned} X_{n+1/2} &= F^T [(I-P)FX_n + Y - f_n] \\ X_{n+1} &= R(U\bar{\sigma}V), \text{ with } [U \quad \sigma \quad V] = SVD([X_{n+1/2} \quad X_0]) \text{ and } \bar{\sigma} = \max(\sigma - 2\lambda, 0) \\ f_{n+1} &= f_n + PFX_{n+1} - Y \end{aligned} . \tag{5}$$

Step 1 in Eq. (5) simply means the k-space comes from the acquired data Y for the sampled k-space and from the last iterate X for the un-sampled k-space, up to a residual consistency term f_n. Steps 2 is to solve the constrained nuclear norm problem (i.e., the second equation of (3)), which mainly consists of the singular value decomposition (SVD) for global sparsity, and then the thresholding for finding the principal components to enforce the self similarity of X and the similarity of X with prior images X_0. Step 3 is to add the uncorrected residual back to be corrected, which is equivalent to decrease λ. Note that the algorithm described by Eq. (5) can also be applied for the LR model when no prior image is available. Unlike the local sparsity model (e.g., the total variation or the wavelet sparsity model), the inclusion of an arbitrary number of prior images is straightforward for PCLR. Moreover, the additional computational cost from LR to PCLR is practically negligible.

The tuning of the regularization parameter is not necessary when the problem is scaled. That is we scale the k-space data so that the maximum of the image magnitude is nearly one. This for example can be efficiently done through a zero-filling Fourier transform. Then we choose $2\lambda=1$. Here as long as λ is sufficiently large (e.g., from 0.1 to 1), the residual update f in (5) (without updating λ) will be equivalent to solving Eq. (2) with a fine-tuned λ. As a result, Eq. (5) is nearly parameter-free. However, an educated guess of λ will certainly accelerate the solution convergence by reducing the number of iterations. On the other hand, the relative difference (i.e., $\|X_{n+1}-X_n\|$) serves as the stopping criterion.

Results: The image reconstruction results from LR without any prior image (LR), PCLR with 2 prior images (LR2), and PCLR with 4 prior images (LR4) are presented in Fig. 1. Here the DW images have 60 DW directions (D=60), SNR=30, various b values (i.e., b=1000, 2000, and 3000), 6-fold and 10-fold k-space undersampling respectively. The prior images were generated with different DW directions from the above 60 directions, with the same SNR and the same b value, to avoid a bias in the prior images. Fig. 1 indicates that PCLR improves LR in terms of both the reconstruction error and the image quality.

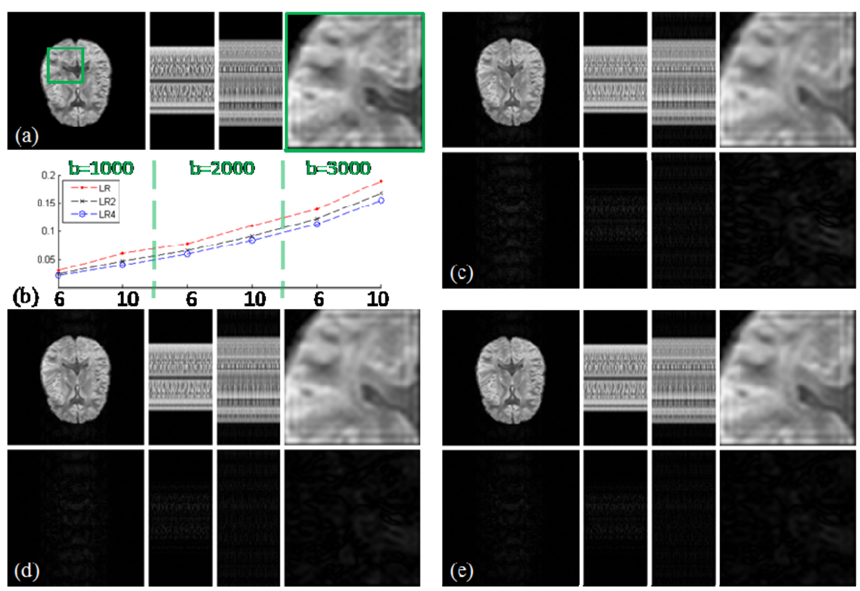

Fig. 1. Image reconstruction via LR and PCLR. (a) The gold standard. (b) The plot of the total reconstruction errors from LR, PCLR with 2 prior images (LR2), PCLR with 4 prior images (LR4), with respect to the undersampling ratio (6, 10) and the b value (1000, 2000, 3000), at D=60 and SNR=30. (c), (d), and (e) are the results from LR, LR2 and LR4 respectively with 10-fold undersampling. For (c)-(e), the 1st row consists of the reconstructed images represented in a DW direction (x-y), the central x-slice with all DW directions (x-q), the central y-slice with all DW directions (y-q), and the zoom-in detail of the ROI; the 2nd row consists of their differences from the original images. Here the prior images are generated with different DW directions from the above 60 directions, with the same SNR and the same b value, to avoid the biased prior images.

Conclusion and Discussion: Since LR is suitable for diffusion MRI, we have introduced PCLR to improve LR when prior images are available. When multiple prior images are available, the use of these priors in the local sparsity model may be tedious, e.g., one may need to additionally enforce the local difference between every image to be reconstructed and every prior image. However, the PCLR method described here is able to utilize many prior images with negligibly increased computational time. In addition, a simple-to-implement and efficient algorithm has been proposed to solve PCLR.