A Quantum Description of Signal Transduction in Magnetic Resonance: Analytical Results, Cavity Damping, and Relaxation

James Tropp[1]
[1]*GE Healthcare Technologies, Fremont, CA, United States*

We have expanded our earlier quantum mechanical treatment (*1, 2*) of the free induction decay and radiation damping, to include some analytical results, and also to incorporate the effects of cavity losses and spin relaxation. Our earlier description is based on the Jaynes-Cummings (J-C) model in quantum optics (*3*), for the coupling of a two level atom (or a spin ½) to a lossless cavity, --in our case, a quantized LC oscillator--; multiple spins may also be accommodated (*4*). Our earlier results were purely numerical, and included no effects of dissipation. We now give some analytic results, and also introduce cavity losses (through a master equation) and spin relaxation (phenomenologically).

The cavity losses are treated by a slight modification of the usual theory for micromasers (*5,6*), which employs only the photon reduced density matrix. Here we use instead the combined spin-photon density matrix, writing the usual cavity operators as Kronecker products with the spin identity. This gives the customary damping term in the Liouville equation:

$$\dot{\rho} = (\gamma/2)\{(n+1)(2\hat{a}\rho\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\rho - \rho\hat{a}^{\dagger}\hat{a}) + n(2\hat{a}^{\dagger}\rho\hat{a} - \hat{a}\hat{a}^{\dagger}\rho - \rho\hat{a}\hat{a}^{\dagger})\}$$

where ρ is the density matrix, the a's are the cavity operators for creation and annihilation (modified as noted), γ is the photon damping rate, and n is the photon occupation number. In addition to this, spin relaxation is introduced through the spin operators for longitudinal and transverse magnetization, allied with appropriate damping constants. The damping rates for both cavity dissipation, and the relaxation (inverses of T_1 and T_2) are scaled arbitrarily to the fundamental frequency of Rabi oscillation (7), here taken as unity, which serves as the temporal reference throughout.

Also, an elementary calculation yields the reduced density matrices at baseband for spin and the cavity in J-C model, which take the forms:

$$\rho^{(spin)}(t) = 1/2\begin{bmatrix} 1+s^2 & c \\ c & c^2 \end{bmatrix} \quad \text{and} \quad \rho^{(cavity)}(t) = 1/2\begin{bmatrix} 1+c^2 & -is \\ is & s^2 \end{bmatrix}$$

with the abbreviations $c = \cos\frac{1}{2}\Omega_0 t$ and $s = \sin\frac{1}{2}\Omega_0 t$, where Ω_0 is the Rabi fundamental frequency. From these it is apparent that the Rabi oscillation of the longitudinal moment occurs at twice the frequency of transverse moment. This is a pure quantum effect, illustrative of the non-classical motion of the Bloch vector, which disappears in the more familiar Maxwell-Bloch equations (8). The imaginary off diagonal elements in the cavity matrix arise from the quadrature relation between transverse magnetization and cavity field at the Larmor frequency. Comparison with the spin density matrix shows as well a quadrature offset at baseband.

The effects of dissipation are illustrated in the figures, which show the effects of progressive increase, starting from a lossless cavity and infinite spin relaxation times at the far left. The spin is prepared with an initial tip of π/2; the transverse moment is shown in green, the longitudinal in blue, the photon population in red, and the total excitations (the deviation of the longitudinal moment from its starting value, plus the number of photons; absent dissipation, a conserved quantity), in dotted black. The time scale is in Rabi periods. Figure 1 shows repetitive undamped Rabi oscillation, continuing indefinitely. Figure two, with mild damping, shows a slow decrement in the amplitudes of the oscillatory quantities, and the total excitations. Figure 3 (strong damping of the cavity and the transverse moment, weak damping of the longitudinal, shows a close resemblance to classical radiation damping, with the transverse moment approaching zero, as the cavity excitation dies out. The longitudinal moment has begun an oscillatory path, but will eventually damp very slowly to zero, due to the long T_1.

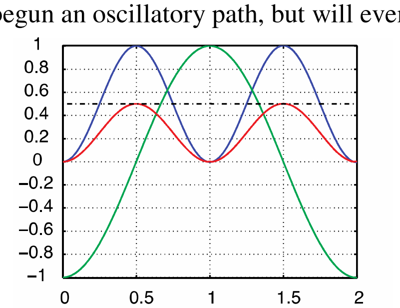

Fig. 1: Time evolution of magnetizations and photon population, with undamped cavity and infinite relaxation times.

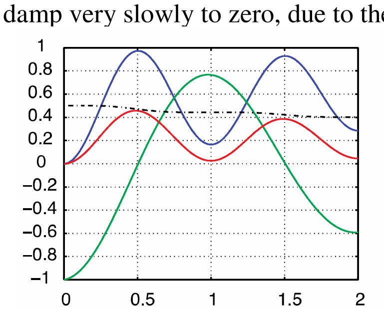

Fig. 2: Similar to Fig. 1 but with moderate damping. Longitudinal and transverse relaxation rates 0.1/sec and 0.2/sec; photon damping rate 0.4/sec.

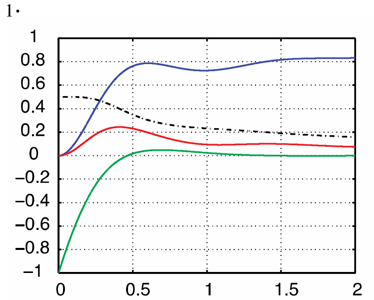

Fig. 3: Similar to Fig. 2 but with extreme damping for photons and transverse magenetization. Longitudial relaxation 0.1/sec; transverse relaxation 4.0/sec, photon damping 5.0/sec.

References:

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