Sampling Density Compensation Function Estimation by Regularized Conjugate Gradient Iteration with a Reduced Oversampling Ratio

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Introduction The iterative convolution based estimation of non-Cartesian sampling density compensation function (DCF) involves a Gridding, an Inverse Gridding and a ratio iterative correction in each loop [1]. An intermediate Cartesian grid typically oversampled by a ratio of two is required to achieve high convergence accuracy and this increases the memory and computational burden heavily when processing data of large size or of high dimension. The iteration converges stably but saturates easily which limits the capacity to achieve higher estimation accuracy. In this work, an iterative procedure based on the Conjugate Gradient (CG) method is introduced to reduce the oversampled grid size while achieving higher DCF estimation accuracy without increasing computational and storage burden. Due to the ill-condition of the linear system for DCF estimation, in each loop the DCF estimated by the original method is employed as a regularization term to form a bi-criterion optimization problem which can be solved by the CG iteration with fast convergence.

Methods Faithful DCF is crucial to the quality of the image reconstructed from non-Cartesian k-space samples by Gridding based algorithms. Based on the ideal PSF criterion, the DCF denoted by w should satisfy the following linear equation:

$$G_{GA}G_{AG}w = u \qquad (1)$$

where u is the all-ones vector with the same length as w, G_{AG} and G_{GA} are the Gridding and Inversing Gridding coefficient matrices that convolve data at arbitrary positions to a Cartesian grid and vice versa. Inherently, Eq. (1) is the matrix representation of $[(w * c) \cdot \mathrm{III}] * c = 1$, where c is the convolution kernel used to compute the entries of G_{AG} and G_{GA}, III is an intermediate Cartesian grid [1]. In [2], an iterative procedure with a novel ratio correction was proposed to find approximate solution to this linear equation where each loop of the iteration process involves a Gridding and an Inverse Gridding operation to compute $G_{AG}w_i$ and $G_{GA}G_{AG}w_i$ sequentially. To achieve high convergence accuracy, the Cartesian grid is required to be oversampled due to the increase of the oversampling ratio would improve the condition number of Eq. (1). In [1], it was reported that an oversampling ratio R should be 2.1 to achieve high convergence accuracy. However, it should be noted that the computational and storage load in each loop is proportional to the oversampling ratio R, and the oversampled gird will increase the storage and the convolution computational burden significantly. In addition, the ideal DCF determined by the trajectory should be irrelevant with the estimation algorithm. Thus, achieve accurate DCF estimation with low oversampling ratio is a challenging problem.

As indicated by (1), the DCF estimation problem is inherently to find the solution to the matrix equation whose condition number and positive definiteness are determined by the non-Cartesian trajectory, the adopted convolution kernel [3], and the oversampling ratio. In this work, in order to solve this problem accurately and efficiently, the CG iteration is adopted because it can be easily introduced with a powerful and robust iteration correction capacity and with almost negligible increment in computation. In addition, to avoid the oscillation introduced by bad condition number, regularization is necessary to be introduced to stabilize the convergence process of CG to accelerate the convergence speed. The regularization employed is to minimize the weighted sum of the objectives:

$$\left\|G_{GA}G_{AG}w^{cg} - u\right\|_2^2 + \gamma \left\|w^{cg} - w^{r}\right\|_2^2 \qquad (2)$$

where $\gamma \geq 0$ is a parameter that trades between the two objectives and set to 1 for simplicity, w^{cg} and w^{r} are the DCFs estimated by the CG and the original method separately. Since the convergence of the original method is smooth and the estimated DCF is positive without constraint, w^{r} is a perfect regularization term and is calculated simultaneously with w^{cg} in each CG loop. Other regularization is also feasible [4]. The proposed CG iteration is listed in Table 1.

$$
\begin{aligned}
& w_0^{cg} = w_0^{r} = \frac{u}{G_{GA}G_{AG}u} \\
& r_0 = u - G_{GA}G_{AG}w_0^{cg} \\
& p = r_0 .* w_0^{r} \\
& \varepsilon = required\ accuracy \\
& for\ \ i = 0, 1, \cdots \{ \\
& \quad if\ \ \frac{r_i^H r_i}{u^H u} < \varepsilon,\ exit;\ else\ \{ \\
& \qquad q^{cg} = G_{GA}G_{AG}p \\
& \qquad q^{r} = G_{GA}G_{AG}w_i^{r} \\
& \qquad w_{i+1}^{cg} = w_i^{cg} + \frac{r_i^H p}{p^H q^{cg}} p \\
& \qquad w_{i+1}^{r} = \frac{w_i^{r}}{q^{r}} \\
& \qquad r_{i+1} = r_i - \frac{r_i^H p}{p^H q^{cg}} q^{cg} \\
& \qquad q^{cg} = r_{i+1} .* w_{i+1}^{r} \\
& \qquad p = q^{cg} + \frac{r_{i+1}^H q^{cg}}{r_i^H p} p \ \ \}\}
\end{aligned}
$$

Table 1. CG with regularization for DCF estimation

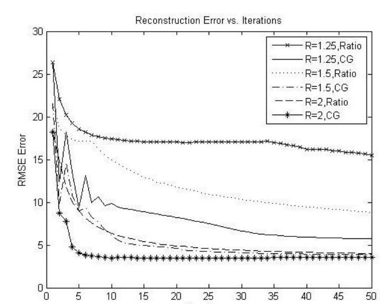

Figure 1. RMS Error convergences of reconstructed images at various levels of oversampling

Results A set of non-Cartesian k-space data along a spiral trajectory was applied for experiment which was provided by MRI Unbound at ISMRM website. The data consisted of 8 interleaves each with 20000 points and was used to reconstruct a 320×320 image. The presampled Kaiser-Bessel kernels which were optimized (all with width of 6) for Gridding k-space data were utilized to synthesize the system matrices under different oversampling ratios [5]. All iterations utilized Jackson's DCF [6] as the initialization or the precondition. The RMS error convergences of the reconstructed images using DCFs estimated by the proposed CG and the original Ratio iterations at various oversampling ratios were depicted in Fig. 1. Using a reduced oversampling ratio of 1.25, the DCF estimated by the proposed algorithm can still offer high convergence accuracy. The final RMS errors after 50 iterations of each method and the RMS errors relate to the non-iterative Jackson's DCFs were plotted for various oversampling ratios R in Fig. 2. It can be found that the improved CG iteration can reduce the impact of the oversampling ratio on the accuracy of estimated DCF.

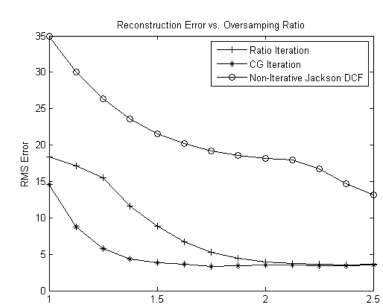

Figure 2. RMS Errors of images after 50 iterations using various algorithms and oversampling factors

Conclusion In this work, the iterative DCF estimation algorithm was improved by solving a bi-criterion optimization problem using the Conjugate Gradient algorithm to reduce the oversampling ratio of the Cartesian grid while achieving higher accuracy and accelerate the convergence speed without increasing computational or storage burden comparing with the original method. Although reducing the oversampling ratio would worsen the condition number of the linear system for DCF estimation, our result demonstrated that the introduction of estimated DCF by Pipe's method in each loop as regularization for the proposed CG iteration can stabilize the convergence efficiently and obtain high convergence accuracy with a much lower oversampling ratio.

References [1] Zwart NR, et al., MRM, 2011; [2] Pipe JG, et al., MRM, 41:179-186, 1999; [3] Johnson KO, et al., MRM, 61:439-447, 2009; [4] Bydder M, et al., MRI, 25:695-702, 2007; [5] Beatty PJ, et al., IEEE TMI, 24:799-808, 2005; [6] Jackson JI, et al., IEEE TMI, 10:473-478, 1991.