Fast l1-minimization for Compressed Sensing using Orthonormal Expansion

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Introduction: Compressed Sensing (CS) enables great acceleration of MRI acquisition through k-space random sampling of highly compressible images [1]. The reconstruction of CS images involves constrained ℓ_1-minimization, for which standard convex optimization technique, such as nonlinear conjugate gradient (NCG) [2], suffer from slow convergence. On the other hand, many iterative thresholding techniques have been developed for computational tractability, but at the expense of accuracy [3]. This work introduces a fast ℓ_1-minimization algorithm for CS based on orthonormal expansion of sensing matrix.

Method: CS reconstruction in MRI is formulated as follows:

$$\mathrm{X} = \arg\min_{\mathrm{X}}\{\|\mathrm{X}\|_1 : \mathcal{F}_\Omega \Psi^{-1}\mathrm{X} = \mathrm{b}\},$$

where X is the sparse representation of the MR image to be reconstructed from the acquired k-space data b. Ψ is the sparsifying transform, and $\mathcal{F}_\Omega$ is the partial Fourier transform over the k-space down-sample subset Ω. Denote $\mathcal{F}_\Omega\Psi^{-1}$ as sensing matrix A, which contains a subset of rows of a unitary matrix $\Phi = [\mathrm{A}^\mathrm{T}\ \mathrm{B}^\mathrm{T}]^\mathrm{T}$. B is the complementary partial orthonormal matrix, whose rows are orthonormal to those of A. By expanding A to Φ, we cast the standard CS ℓ_1-minimization into the following equivalent form:

$$\min_{(X,p)} \|X\|_1 \quad \text{subject to } \Phi X = p, P_\Omega(p)=\mathrm{b},$$

where $P_\Omega(p)$ is an operator projecting the vector p, the nominal fully-sampled k-space data, onto the down-sample subset Ω. The new formulation can be written into the following augmented Lagrangian function:

$$L(X,p,y,\mu) = \|X\|_1 + \frac{\mu}{2}\|p - \Phi X + \mu^{-1}y\|_2^2 - \|y\|_2^2/2\mu,$$

where μ is a positive scalar and y is the Augmented Lagrange Multiplier (ALM) [4] that introduces an additional quadratic penalty term. This augmented Lagrangian function is minimized by alternatively solving two sub-problems:

(1): $\min_X L(X,p,y,\mu)$ and (2): $\min_{p,P_\Omega(p)=\mathrm{b}} L(X,p,y,\mu)$.

It is the orthonormal expansion of the sensing matrix A to Φ that enables the first sub-problem to be optimally solved with a soft-thresholding operation; i.e., $\arg\min_X L = S_{\mu^{-1}}\left(\Phi^{-1}(p + \mu^{-1}y)\right)$, where $S_{\mu^{-1}}(w) = \mathrm{sgn}(w)(|w| - \mu^{-1})^+$. The second sub-problem is solved by letting $\frac{\partial L}{\partial P_{\bar\Omega}(p)} = 0$, where $\bar\Omega$ denotes the complementary of the sub-sample set Ω. The solution to the second sub-problem is $P_{\bar\Omega}(p) = P_{\bar\Omega}(\Phi X - \mu^{-1}y)$, and $P_\Omega(p)$=b is enforced as a data-fixing operation.

The proposed algorithm outlined in Exhibit 1 was evaluated with both Shepp-Logan phantom and a brain study. The phantom and brain images underwent 85% and 67% down-sampling with a Poisson-disk random sampling pattern [3]. The reconstruction performance is compared with the NCG method as described in [2]. Both algorithms were implemented in MATLAB, running on a Linux PC equipped with 3 GHz Intel Core2 DUO CPU and 3 GB memory.

Exhibit 1: Outline of Fast ℓ_1-minimization

Input: $Y_0 = 0, \mu_0 > 0, b, \Omega, \Phi, p_0 = \begin{bmatrix} b \\ 0 \end{bmatrix}$;
While not convergent **do**
//Step 1 solves $x_{k+1} = \arg\min_x L(x, p_k, Y_k, \mu_k)$
Step 1: $x_{k+1} = S_{\mu_k^{-1}}(\Phi^{-1}(p_k + \frac{1}{\mu_k}Y_k))$
//Step 2 solves $p_{k+1} = \arg\min_p L(p, x_{k+1}, Y_k, \mu_k)$
Step 2: $p_{k+1} = \begin{bmatrix} b \\ P_{\bar\Omega}(\Phi x_{k+1} - \mu_k^{-1}Y_k) \end{bmatrix}$
Step 3: $Y_{k+1} = Y_k + \mu_k(p_{k+1} - \Phi x_{k+1})$
Step 4: $\mu_{k+1} > \mu_k$
Step 5: $k = k + 1$
end
Output: x_k, p_k

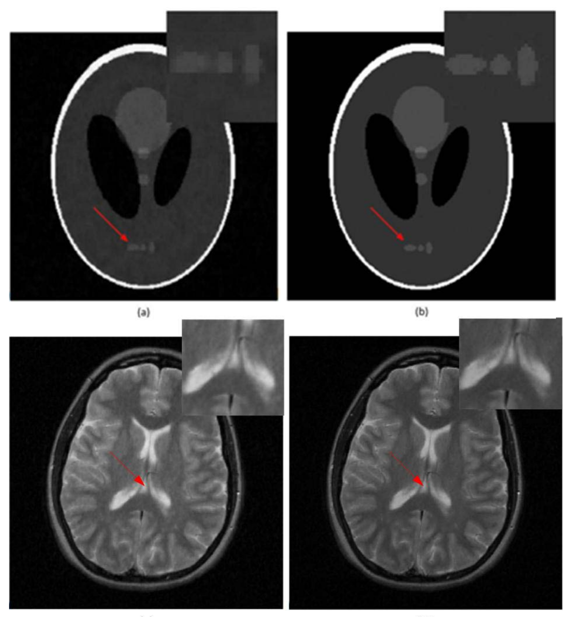

Fig. 1: The proposed algorithm achieves significantly fast convergence than the NCG method. The phantom and brain images obtained from the proposed algorithm (b, d) are more accurate than that of NCG method (a, c), as manifest with better depiction of image details.

Result: Fig. 1 compares the phantom and brain reconstruction results after running the NCG method and the proposed algorithm for 50 iterations. For the phantom study, the NCG method and the proposed method resulted in 8.3% and 1.21% Root-mean-squares (RMS) errors, respectively. For the brain study, the RMS error of the proposed method is 2.29%, which is lower than 5.67% RMS error of NCG.

Conclusion: Orthonormal expansion of the sensing matrix into unitary enables fast ℓ_1-minimization: soft-thresholding that incurs little computational cost optimally solves ones of the sub-problems in each iteration. The proposed method holds great potential for accurate real-time on-line CS reconstruction.

References: [1] Donoho, IEEE Info. Theory 2006:(52):1289-1306. [2] Lustig et al. MRM 2007:(58):1182-95. [3] Maleki et al. IEEE Sig. Proc. 2010:(4):330-41. [4] Nocedal Numerical Optimization, Springer, 1999