Fast Approximators for Least-Norm Reconstructions of Undersampled Non-Cartesian MRI Data

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Introduction: Least-norm reconstruction of undersampled Cartesian MRI data, commonly referred to a "zero filling" [1], remains a popular strategy due to its simple and efficient implementation, and readily-characterized behavior. However, despite its mathematical simplicity, least-norm reconstruction of undersampled non-Cartesian MRI data is computationally intensive, requiring use of iterative methods that may be impractical for clinical use [2]. Here, we describe a novel and efficient numerical framework for generating accurate, non-iterative (i.e., direct) approximators of least-norm reconstructions of non-Cartesian MRI data.

Theory: Suppose we model our signal acquisition as $g = Hf$, where H is the $N \rightarrow K$ discrete-time Fourier transform (DTFT) and f is the (discrete approximation of the) image of interest. To mitigate rank-deficiency challenges, we can consider the equivalent system, $Rg = RHf$, where R "prunes" redundant rows of H and g (via averaging) such that RH is full row rank. When $K<N$, the problem of recovering f from g is underdetermined. A standard approach for such scenarios is least-norm estimation,

$$\hat{u} = \arg\min_{\substack{u \in C^N \\ Hu=g}} \|u\|_2^2 = H^* R^* \left(RHH^* R^*\right)^{-1} Rg \qquad (1)$$

For Cartesian imaging, the crosstalk matrix is the identity scaled by a constant and factors out of (1), yielding "zero filling." For non-Cartesian, however, this is not the case, and inversion of the crosstalk matrix is computationally challenging. Thus, we approximate it by a diagonal matrix, D=diag(d), so as to permit a direct reconstruction, namely $\hat{u} \approx H^* R^* DRg$ [3-5]. Since $(RHH^*R^*)^{-1}$ is positive definite, D should be as well; and so d should be real and non-negative. Adopting a least squares loss function, the (constrained) least norm approximator design problem is then

$$\hat{d} = \arg\min_{\substack{d \in R^K \\ d \geq 0}} \left\| R^* H^* \left(RHH^* R^*\right)^{-1} Rg - H^* R^* DRg \right\|_2^2 = \arg\min_{\substack{d \in R^K \\ d \geq 0}} J(d) \qquad (2)$$

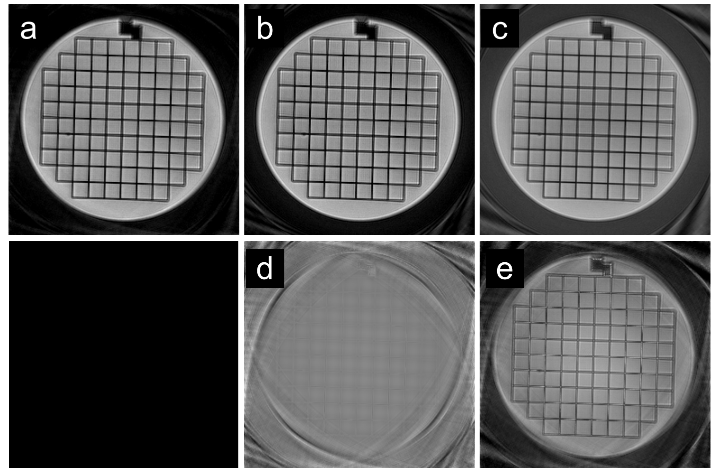

FIG 1) Example reconstructions from an Archimedean spiral sequence using (a) exact least norm estimation, (b) approximate least norm estimation, and (c) gridding with density compensation [9]. Images were normalized to have the same mean value, and identically windowed and leveled. (d) and (e) are magnitude difference images between (b) and (c), and the exact least-norm estimate, respectively. Off resonance correction was not performed. Note that the proposed least norm approximation closely matches the true solution.

Ideally, g would be simultaneously optimized to maximize J, akin to the construction in [6] for gridding kernel optimization; however, this would resort (2) to spectral norm optimization which is computationally impractical. Instead, we employ $g=e$, the unit vector and expected spectral response from sampling of a delta function. The gradient of (2) under $g=e$ is $\nabla J(d)=2(RHH^*R^*d-e)$. As (2) is convex, in this work we adopt Nesterov's proximal gradient method [7] for solving this problem, which is defined as follows:

$$\textbf{INIT:}\ d_0 = 0, t_0 = 1, y_0 = d_0 \qquad \textbf{REPEAT:}\ 1)\ d_{k+1} = P\left(y_k - \tfrac{1}{L}\nabla J(y_k)\right) \quad 2)\ t_{k+1} = \tfrac{1}{2}\left(1+\sqrt{4t_k^2+1}\right) \quad 3)\ y_{k+1} = d_{k+1} + \frac{t_k - 1}{t_{k+1}}(d_{k+1} - d_k)$$

where $P()$ is the non-negative real projector and the Lipschitz constant, L, can be determined via power iteration. Note that the positive-definiteness of RHH^*R^* only asserts d^*e=Re$\{d^*e\}$>0, and so the solution to $\nabla J(d)=0$ and (2) are not necessarily equivalent. Fortunately, Nesterov's scheme provides a very efficient mechanism for determining a solution to the constrained optimization problem. Interestingly, despite possessing a different mathematical objective, the proposed least norm approximator is semantically similar to several existing methods for direct non-Cartesian reconstruction. For example, several authors have considered solving $\nabla J(d)=0$ (and closely-related forms) based on point-spread function (PSF) optimization arguments [8-10]. Also based on PSF arguments, Samsonov et al. [11] suggested a non-negative projected gradient descent that is related to our approach for solving (2); and Bydder et al. [12] later incorporated an added fidelity term (to Jackson et al.'s [8] estimate) that encourages, but does not explicitly enforce, non-negativity.

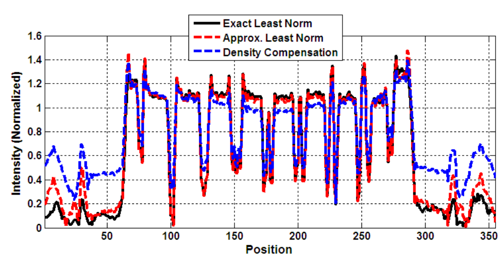

FIG 2) Line profiles from top-left to bottom-right corners of images (a)-(c) in Fig. 1. Note that the propose least norm approximator closely matches the exact solution. At the same time, the estimate of Ref. 9 exhibits greater intensity non-uniformity.

Example: Fig. 1 shows example 256x256 reconstructions of a 16-shot Archimedean spiral acquisition (single-channel, 4096 samples/shot, repeat sampling at the origin→underdetermined) of a resolution phantom obtained by directly solving (1), using the proposed approximation in (2), and, for reference, the iterative method of Ref. [9]. A 1.125x oversampled non-uniform FFT (NUFFT) [6] employing a W=6 Kaiser-Bessel kernel [13] was used to realize H. Equation (1) was solved via 50 conjugate gradient iterations, which required 0.7176s by a multithreaded C++ implementation (OpenMP,FFTW) running on dual 6-core 3.0GHz machine. Conversely, direct non-Cartesian reconstruction via adjoint NUFFT requires ~0.008 s on the same machine, which is almost 90x faster. Estimation of d according to (2), via 50 iterations of Nesterov's algorithm, required only 0.8163 s of computation, and yet can be reused for later reconstructions. Similarly, 50 iterations of the method in Ref. 9 required only 0.4024s.

Summary: We have proposed a novel and efficient numerical strategy for generating fast and accurate approximators of least-norm reconstructions of undersampled non-Cartesian MRI data. Beyond standalone application, the proposed method can also be used to reduce the computational complexity of iterative non-Cartesian reconstruction methods requiring repeated calculation of least-norm estimates, such as equality-constrained Compressive Sensing strategies [14].

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