Gridding: Exploring an Efficient Numerical Algorithm for 0-th Order Prolate Spheroidal Wave Function Evaluation as Convolution Kernel

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Introduction

In MRI, Fourier reconstruction from non-uniform sampling schemes is performed classically by gridding[1,2], for which the reference convolution kernel is the Kaiser-Bessel window. The Kaiser-Bessel (K-B) window was originally introduced[3,4] as an efficient approximation to the 0^{th} order prolate spheroidal wave function (PSWF). The PSWFs were themselves developed[5] in the analysis of band-limited functions, and the 0^{th}-order PSWF ($PSWF_0$) features the interesting property of best concentrating energy both in the signal and the signal Fourier transform domains. For numerical evaluation purpose, they had to be computed by lengthy processes[4], for instance iteratively involving Fourier transformation at each step[2]. Recently, new algorithms for the computation of PSWFs were introduced and we explore here the efficient computation of 0^{th}-order PSWFs using the algorithm described in [6] as a possible alternative to the Kaiser-Bessel window for gridding reconstruction.

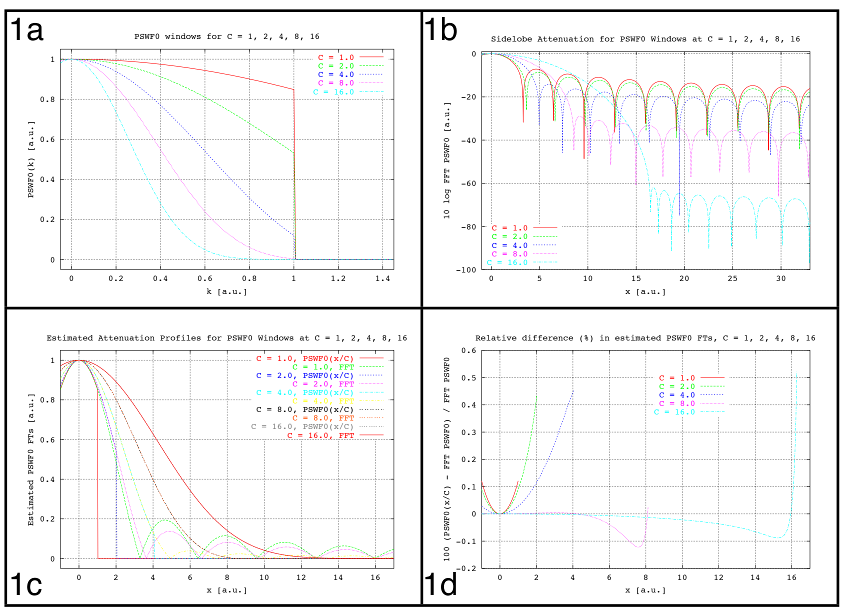
Material and methods

We implemented in C the algorithm described by Xiao et al.[6] to compute the $PSWF_0$ within a specified accuracy ε by means of a truncated series $\sum p_k P_k(x)$ of Legendre polynomials $P_k(x)$. We used LAPack[7] subroutine *dpteqr()* (from CLAPack version 3.2.1) to determine the eigen-vector which components give the coefficients weighting the Legendre polynomials in the approximating series. Due to the symmetry of the 0^{th}-order PSWF, only even orders appear. For evaluation of the series, the Legendre polynomials were estimated from the well-known 3-terms recurrence relation[8]: $(k+1)\,P_{k+1}(x) = (2k+1)\,x\,P_k(x) - k\,P_{k-1}(x)$, with $P_0(x) = 1$ and $P_1(x) = x$.

To allow application of the Fast Fourier Transform (FFT) algorithm for Fourier reconstruction, gridding uses convolution of the non-uniformly distributed discrete samples with a continuous kernel to allow resampling the resulting continuous convolution product at uniformly distributed nodes of a cartesian grid. A necessary step after FFT is to correct the obtained image for the weighting effect associated with the Fourier transform (FT) of the convolution kernel. In our case, the kernel is a finite sum of weighted Legendre polynomials, and its FT could be obtained as the correspondingly weighted finite sum of spherical Bessel functions of first kind, which are FT of the Legendre polynomials. In our current implementation however, we have opted instead to use the property of the $PSWF_0$ of bandwidth C to be its own FT after argument stretching, $FT[\,PSWF_0(k;C)\,] = \lambda_0\, PSWF_0(x/C;C)$, and to trust the truncated Legendre polynomial series approximation to reasonably well reproduce this property. Numerical checks were performed, comparing the FFT (using FFTW 3.2.2[9]) obtained from computed samples of the $PSWF_0$ to computed samples of the appropriately stretched $PSWF_0$ approximation of same bandwidth. Examples are shown in Fig. 1c and 1d below.

Results

For the displayed results, we worked with a specified accuracy $ε = 10^{-12}$. In Fig. 1a, we represent the $PSWF_0$s of bandwidth C = 1, 2, 4, 8, 16, and in Fig. 1b their FT obtained by FFT computation. In Fig. 1c, we compare the attenuation profiles to be corrected for after FFT estimated both by FFT of the data used for Fig. 1a and by computing the $PSWF_0$s of same bandwidths with stretched argument. Fig. 1d represents the difference of the latter relative to the FFT computed reference.



Discussion

It is worth noting that as soon as the bandwidth parameter C and the accuracy target ε are set, the eigenproblem is specified and needs to be numerically solved just once. Evaluating the approximating series for a new argument only implies evaluating the Legendre polynomials by the recurrence relation, and accumulating the weighted sum using the stored coefficients: basic computational operations which do not involve calls to any special or external function, nor conditional tests. Evaluating the Kaiser-Bessel kernel normally implies call to the Bessel I0 and to the square root functions, if not implemented using a look-up table approach.

The Fourier transform of the $PSWF_0$ approximation estimated from the stretched $PSWF_0$ is very close to that computed by FFT of $PSWF_0$ samples. Both approaches seem viable alternatives for the estimation of the correction of reconstructed profiles for convolution effects. Sampling the stretched $PSWF_0$ features using a single function with same bandwidth and accuracy parameters for both the convolution kernel and its profile after Fourier transformation.

Conclusion

We have shown that thanks to recently developed algorithms it is now possible and easy to compute the 0^{th}-order PSWF as convolution kernel for gridding reconstruction, offering an interesting alternative to using its Kaiser-Bessel approximation.

We tested two algorithms to compute the post Fourier transformation weighting necessary to compensate for multiplication profile effects in image space associated with convolution in k-space: either using a FFT or evaluating the $PSWF_0$ with appropriately scaled argument.

References

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