Towards real-time 4D field shift predictions: optimizing Fourier-based calculations of the susceptibility induced perturbation of the magnetic field

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Introduction:
A fast and accurate forward calculation of the susceptibility induced field shift is important in quantitative susceptibility mapping, MR simulations, real-time shimming, real-time interventions and MR thermometry. Fourier-Based (FB) methods [1,2] fulfil these requirements, however, suffer from inherent (wrap-around) aliasing due to the periodic decomposition. An accurate calculation requires a spatial padding factor of two in all three dimensions, due to the infinite magnetic dipole [1,2]. Since FB-calculations scale with $O(N^3 \log_2(N^3))$, the time needed increases eightfold, severely hamperering the feasibility of dynamic field calculations. However, this seems disproportional regarding the smoothness of aliasing.
Herein is proposed to decouple the calculation procedure into two separate parts: the convolution of the 'unpadded' Field Of View (FOV), and an additional convolution to calculate the effect of aliasing, which can be used to correct the initial outcome. Although this 'virtual' zero-padding is mathematically equivalent to conventional padding, decoupling has three key advantages: 1) calculation may be about four times faster, depending on the resolution in which the aliasing is estimated. 2) objects of interest can be eight times larger than before, as alias-prevention is dealt with afterwards 3) both parts of the calculation can be updated separately, allowing real-time field applications. Moreover updating the internal field can be prioritized, since alias correction may have a prolonged validity. The only drawback lies in negligible accuracy loss in the periphery of the FOV.

Theory: conventional zero-padding
Convolution is allowed if susceptibilities do not exceed 100 ppm[1], and is efficiently calculated by a multiplication in the frequency domain: $\delta_{spoiled}(\vec{r}) = F^{-1}[F(\chi(\vec{r}) - \chi_{air}) \times G(\vec{k})]$, however the resulting relative field shift is spoiled by aliasing without proper padding. F and F^{-1} are the forward and inverse Fourier Transform, $\chi(\vec{r})$ is the susceptibility distribution, $\chi_{air} \approx .36$ ppm and $G(\vec{k})$ is Green's function in the frequency domain. To minimize aliasing, the FOV must be centrally embedded in a large empty matrix, an operation called Zero-Padding (*Z*), and the field induced in this spatial buffer is Cropped (C) after calculation: $\delta_{cleared}(\vec{r}) = C\{F^{-1}[F(Z\{\Delta\chi(\vec{r})\}) \times G(\vec{k})]\}$.

Methods: virtual zero-padding:
The low-res aliasing used to correct the initial hi-res convolution Eq.[1], was estimated by additional convolutions of downscaled versions of the original distributions Eq.[2]. This general formulation was separately optimized for two different types of FB-convolution, viz. circular convolution (CC) Eq.[3] and k-space filtering (KF) Eq.[4], distinguished by their kernel, being either based in the spatial domain (CC: Eq.[5]) or in the frequency domain (KF: Eq[6]).

$$\delta_{cleared} \approx \delta_{spoiled} - \delta_{error} \quad \text{Eq.[1]}$$

$$\hat{\delta}_{error} = U(F^{-1}[F(D(\Delta\chi)) \times G(\vec{k})] - C\{F^{-1}[F(Z\{D(\Delta\chi)\}) \times G(\vec{k})]\}) \quad \text{Eq.[2]}$$

$$\hat{\delta}^{CC}_{error} = U \cdot C[F^{-1}(F\{Z[D(X)]\} \times F\{G(\vec{r})\}) - R(F\{D(X)\} \times F\{G(\vec{r})\}))] \quad \text{Eq.[3]}$$

$$\hat{\delta}^{KF}_{error} = U \cdot C[F^{-1}(F\{Z^c[D(X)]\} \times G(\vec{k}))] \quad \text{Eq.[4]}$$

$$G(\vec{r}) = \begin{cases} \frac{1}{4\pi} \times \frac{3z^2 - r^2}{r^5} & if \quad \|\vec{r}\| \neq 0 \\ 0 & if \quad \|\vec{r}\| = 0 \end{cases} \quad \text{Eq.[5]}$$

$$G(\vec{k}) = \begin{cases} \frac{1}{3} - \frac{k_z^2}{k_x^2 + k_y^2 + k_z^2} & if \quad \|\vec{k}\| \neq 0 \\ 0 & if \quad \|\vec{k}\| = 0 \end{cases} \quad \text{Eq.[6]}$$

R is the repetition operator, similar to *Z*, however the central FOV is now surrounded by copies. Z^c is complementary to Z: $Z^C(X)=R(X)-Z(X)$.
Down- and up-scaling (*D, U)* were implemented in the spatial domain. (Both methods include the Lorentz correction [2])
Validation: Procedures were validated with conventional zero-padding, the analytical solution and MR experiments.The capacity of handling large objects without aliasing artifacts was numerically tested by generating a set of spheres of increasing diameters {64,128,.,512} within a matrix of 512^3. To demonstrate the applicability of virtual zero-padding in dynamical field calculations, we simulated how a continuously moving applicator influenced a certain region of interest (ROI). This applicator was modelled by a small sphere, the ROI by a larger sphere. A similar set-up was used for the experimental validation: the field was shimmed on the larger sphere, before the smaller perturber was introduced. Field maps were determined with a 3D gradient echo sequence acquired with multiple echoes.

Results:
Virtual zero-padding reduced calculation time (relative to conventional padding) with ~70% for halving the alias resolution, and with ~80% if resolution is halved again. In Fig,1a,b it can be seen that the outcome of virtual zero-padding coincides with the analytical solution (mean absolute error=.001 ppm), and both are validated by the experimental data. The field is given over two perpendicular profiles within the coronal plane, in the situation that applicator is in the axial position, maximally perturbing the initial homogeneous field in the ROI. The dotted line represents the calculation without alias correction. Fig 1c shows that additional low resolution alias subtraction enables to use the full capacity of the Fourier Transform. Fig.1d demonstrates the superiority of virtual zero-padding in dynamical field calculations: before conventional padding is finished, the outcome is already outdated.

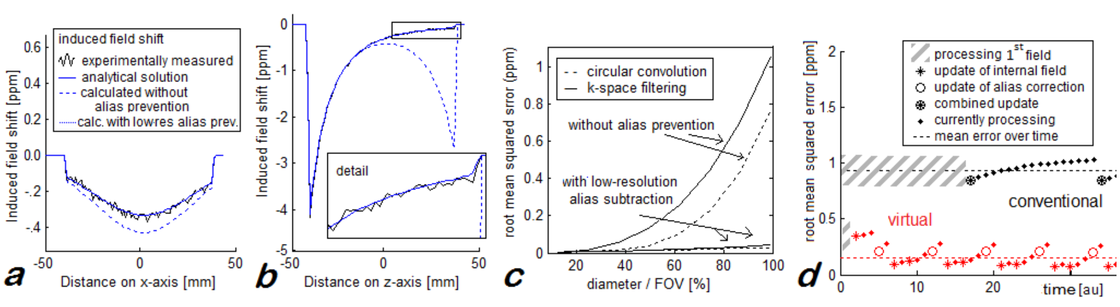

Fig 1a and b Experimental and analytical validation of virtual zero-padding. 1c If the size of an object approximates the capacity of the used Fourier Transform, conventional padding is infeasible, while virtual padding still yields very accurate results. 1d An efficient alternating updating scheme based on virtual zero-padding allows real-time field calculations, whereas conventional padding yields outdated results.

Conclusion:
Virtual zero-padding makes Fourier-based convolutions more than three times as fast, enables to process matrices eight times larger than before, and is superior in real-time field shift predictions. Minor accuracy loss is only seen at the border of the field of view, but is negligible for practical purposes.

References:
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