On density compensation in Bayesian k-space trajectory optimization

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Introduction

In the field of Compressive Sensing (CS) MRI reconstruction, a lot of research has gone into improving reconstruction techniques. Even though many authors initially used (uniformly) randomized sampling patterns, it was experimentally found, and proven in [1], that variable density trajectories have vastly better performance. From a Bayesian viewpoint of CS MRI, an inventive technique was recently proposed in [2] to optimally construct a k-space sampling pattern from a set of different k-space segments, using variational Bayesian approximation. In this work, we will first demonstrate how different similarly subsampled k-space trajectories give rise to vastly different reconstruction results. Then, we show that using a trajectory optimization technique, such as the one in [2], to optimize variable density trajectories requires special care. More specifically, we will show that correctly adjusting for the non-uniform sampling density, using a proper density compensation function (DCF) is important and easily overlooked.

Method

In [2], CS MRI reconstruction is looked at from its Bayesian interpretation. The familiar unconstrained reconstruction formulation is derived as an MAP estimator under Laplacian image prior distribution and Gaussian likelihood distribution. In this way an image x is found as:

$$\hat{x} = \arg\min_x -\log \mathrm{P}\left(y|x\right) - \log \mathrm{P}\left(x\right) \qquad \hat{x} = \arg\min_x \left\|Fx - y\right\|_2^2 + \lambda \left\|Sx\right\|_1$$

Where S is a sparsifying transform and F is the k-space sampling operator. Our algorithm uses the shearlet transformation in this respect [3], because of its optimal sparsifying property. In [2], a strategy for trajectory optimization is proposed. Iteratively, k-space observations y_n are evaluated, from a set of candidate positions, such that the information gain with respect to the current observations y:

$$\mathrm{I}\left(y_n\right) = \mathrm{H}\left[\mathrm{P}\left(x|y\right)\right] - \mathrm{E}\left[\mathrm{H}\left[\mathrm{P}\left(x|y, y_n\right)\right]\right]$$

is maximized. In order to make calculation of the entropy in this framework tractable, a variational Bayesian technique is used to approximate the posterior distribution as a Gaussian distribution of the image variable x: $\mathrm{P}\left(x|y\right) \approx \mathrm{N}\left(x_m; \Sigma^{-1}\right)$ where $\Sigma = F^H F + S^H \Gamma^{-1} S$

where Γ is a diagonal matrix with variances that arises when lower bounding a Laplace distribution by a more tractable Gaussian distribution [2]. It is important to note that in this approximation, it is implicitly assumed that:

$$FF^H = 1 \qquad (1)$$

As the conditional distribution P(y|x) was approximated as:

$$-\log \mathrm{P}\left(y|x\right) = \left\|Fx - y\right\|_2^2 \approx \left(x - F^H y\right)^H F^H F\left(x - F^H y\right)$$

While condition (1) is fulfilled for many experiments, such as subsampled Nyquist rate Cartesian k-space grids, it should be stressed that this is not true in general variable density samplings. In fact, this requirement is exactly the same as the one in classical regridding reconstruction [4-5]. We propose a fast DCF estimation technique to estimate the DCF weights d. It can be proven that optimal density compensation for image reconstruction in a least-squares sense, for a Dirac delta image, is equivalent with:

$$d = \arg\min_d \left\|FF^H d - 1\right\|^2$$

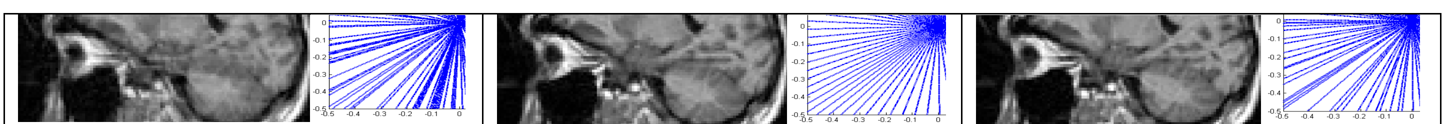
as a delta function has a white spectrum. More interestingly, this formulation also optimizes the requirement (1) in a least squares sense, for a white spectrum. The resulting optimization problem is efficiently solvable with just a few iterations of a conjugate gradient algorithm. Subsequently, this allows for very fast candidate evaluation in the proposed non-uniform trajectory optimization framework.

Experiments and conclusion

Reconstruction of several non-uniformly sampled k-space trajectories was simulated on an anatomical brain image. Due to space constraints, we only provide a peak signal to noise ration PSNR comparison for the reconstruction result. All trajectories were subsampled to 25% of the Nyquist rate and reconstructed using the technique from [3]:

Trajectory type	Regularly subsampled	Random	Random (w/ center)	Radial	Quadratic Spiral	Archimedean Spiral
PSNR	15 dB	18 dB	24 dB	28 dB	33 dB	35 dB

It is clear that optimizing a k-space trajectory can lead to vastly different reconstruction results, which is the main rationale behind our proposed non-uniformly sampled trajectory optimization for CS-MRI. Choosing a suitable k-space trajectory should be the primary concern in CS-MRI applications. In a second experiment, radial sampling was optimized, 25% sub-Nyquist radial sampling of a test image was optimized, choosing the radial lines from an oversampled set of uniformly spaced angles. The results are shown in figure 2.



Random selection of radial lines, PSNR = 26.0dB	Uniform angular spread of radial lines, PSNR = 30.3dB	Optimized selection of lines using proposed technique, PSNR = 31.2dB

Figure 2: Reconstruction results (left) and one quadrant of the k-space (right) of different selections of k-space radial lines from an oversampled candidate set.

The results show that the optimization framework succeeds in achieving improved reconstruction performance in this trajectory optimization experiment.

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