Optimal ordering of diffusion MRI measurements: An extremely efficient and effective approach

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INTRODUCTION Relative merits of various diffusion gradient sampling schemes have been previously investigated (1-4), which indicated that the uniformity of the diffusion gradient directions plays an important role in the quality of any dMRI- or tensor-derived quantities. It has also been shown that different orderings (or sequences) of the gradient directions have different effects on the quality of tensor-derived quantities (5,6) obtained from a partially completed scan. Even though several methods (5-7) have been proposed to generate optimal orderings of gradient directions, the fact that these methods have not been in routine clinical use may be attributed to two major problems of computational inefficiency related to the methods (1,6) for generating uniform and antipodally symmetric points on the unit sphere and the methods (5,6) for generating optimal ordering of gradient directions. The method of Cook et al. (6). took about 22 hours to generate the ordering of 60 points (according to our own assessment), and 137 hours for a set of 150 points (according to Deriche et al. (7)). Deriche et al. (7) proposed an alternative approach by substantially cutting down the computational cost but at the expense of uniformity of the intermediate subsets and the final point set. Here we propose two extremely efficient and deterministic methods for generating and optimally ordering N numbers of gradient directions with computational complexity of $O(N^2)$, i.e., N^2 operations.

METHODS Several deterministic schemes for generating highly uniform points on the unit sphere have been proposed, (8-11), but none of these point sets is endowed with antipodal symmetry. For convenience and simplicity, our proposed method for generating antipodal point set is shown in a compact form in Figure 1. The motivation and geometric intuition behind the method can be found in (12,13). The basic notion of optimal ordering can formulated as a minimization procedure; that is, to obtain subsets of incremental sample size, i.e., two-pair subset, three-pair subset and so on until the full set, that are the best in terms of electrostatic energy among their respective classes. Let $\mathbf{r}_{i1} = -\mathbf{r}_{i2}$ (these are points on the unit sphere), with $i \in \{1,\cdots,N\}$, then $m_i \equiv 1/\|\mathbf{r}_{i1}-\mathbf{r}_{i2}\| = 1/2$. The "electrostatic" energy, denoted by Ω, between two pairs of points, $(\mathbf{r}_{i1},\mathbf{r}_{i2})$ and $(\mathbf{r}_{j1},\mathbf{r}_{j2})$, is given by $\Omega_{ij} = m_i + C_{ij} + m_j$, where is the energy of the cross-term, which is given by $C_{ij} = 2/\|\mathbf{r}_{i1}-\mathbf{r}_{j1}\| + 2/\left(4-\|\mathbf{r}_{i1}-\mathbf{r}_{j1}\|^2\right)^{1/2}$. It is assumed that the first pair has been selected and the first step of our algorithm is to find the best two-pair subset. We proposed the use of an N-dimensional array, Σ_i, to keep track of all the updates in energy computation and the first array is $\Sigma_0 = [m_1, m_2, \cdots, m_N]$. To find the best two-pair, we first update Σ_0 to obtain Σ_1 by adding the cross-terms, C_{1j} from $j=2,\cdots,N$ as follows $\Sigma_1 = [m_1, m_2+C_{12}, m_3+C_{13}, \cdots, m_N+C_{1N}]$, and we then locate the lowest energy term in Σ_1 from the second slot onward. Then, we swap the lowest energy term and its corresponding pair of points with the second term in Σ_1 and the second pair of points, respectively. It is clear that the above procedure can be continued to find larger and larger subsets that are the best among their respective classes until we reach the final Σ, which is Σ_{N-1}.

N = number of gradient directions
n = number of latitudes on the upper hemisphere

1. $n = [\,(N\pi/8)^{1/2}\,]$, [x] denotes integer part of x,

2. $k_i = \begin{cases} \left[\, \dfrac{2\sin(\theta_i)}{\csc(\pi/(4n))}\, N \,\right], & 1 \leqslant i < n \\ N - \sum_{j=1}^{n-1} k_j, & i = n \end{cases}$

3. $\theta_i = \pi\,(i-1/2)/(2n), \quad i = 1, \ldots, n,$
$\phi_{ij} = 2\pi\,(j-1/2)/k_i, \quad j = 1, \ldots, k_i.$

Figure 1. Necessary parameters for generating

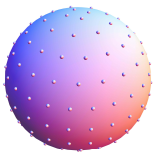

Figure 2. An illustration of the proposed method with 200 points on the sphere.

RESULTS To compare our ordering scheme with those of Dubois et al.(5) and Cook et al.(6), we used their respective orderings of 60 points. Figure 3 shows the comparison of these three ordering schemes. The electrostatic energy of the each subset of size *p* (equivalently, if we consider both the point set and its antipodal counterpart then the actual size used for computing the electrostatic energy is *2p*) is normalized with respect to the electrostatic energy of the point set of size *2p* generated by our previous analytically exact spiral scheme (11). Figure 4 shows the performance of the proposed method in terms of execution time in seconds. For the case of the 60-point set and the 150-point set, the execution time was about 10 ms on both cases. We also evaluated the performance of the method of Cook et al. on the 60-point set and the method took about 22 hours. Therefore, the proposed method was about six orders of magnitude faster than the method of Cook et al. The performance evaluations above were done on a laptop with Intel Core i7 CPU at 1.73 GHz. We should note that the minor variation in execution times seen in Figure 5 may be due to some features of CPU optimization such as dynamic overclocking that are inherent in Intel Core i7 processors. Nevertheless, it is important to note the curve of the execution time as a function of the number of points follows the $O(n^2)$ trend.

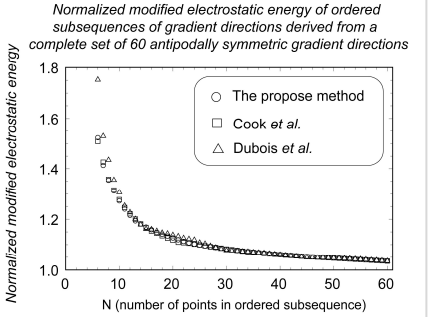

Figure 3. Comparison of normalized energy of various ordering schemes.

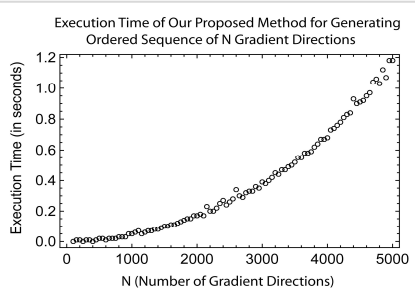

Figure 4. Execution time of the proposed schemes as a function of the number of gradient directions.

DISCUSSION The removal of these major hurdles entails a shift in both our approach for generating the spherical point set that is endowed with antipodal symmetry and our approach for ordering the diffusion gradient directions. Our work shows that it is not necessary to resort to heuristic approaches to solve the problem of optimal ordering of diffusion MRI measurements. Our approach is of comparable or even better performance in terms of uniformity compared to existing methods. Finally, our approach is applicable to 3D radial MRI.

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