Compressed Sensing and HYPR

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Introduction

The desire to achieve high spatial and/or temporal resolution in MRI coupled with limited scan time has led to the necessity to reconstruct images from incomplete datasets. Mathematically this amounts to solving an underdetermined system of equations, that is, the number of unknowns (image pixel values) is larger than the number of equations (acquired measurements). Underdetermined systems of equations generally have an infinite number of possible solutions. A standard approach to isolate a single feasible solution is to incorporate additional prior information about the problem in order to regularize reconstruction and to account for unsampled data points. A number of methods to accelerate MR imaging have been proposed, including parallel imaging [1-3], UNFOLD [4], *k-t* BLAST/SENSE [5], and, more recently, compressed sensing [6-8] and HYPR [9, 10]. We will discuss several different ways to regularize image reconstruction from incomplete data, using both theoretical assumptions and image-specific constraints.

Regularized Image Reconstruction

Mathematical concept of a *norm* is one of the key elements in reconstruction algorithms, especially, the l_p norm defined by

$$\|x\|_p = (\sum |x_k|^p)^{1/p}.$$

Of particular interest in image reconstructions are l_2 , l_1, and l_0 norms (although the last one technically is not a norm as it does not satisfy all the necessary axioms).

MRI signal equation can be represented in the following form:

$$Ef = b, \tag{1}$$

where E is the encoding matrix containing Fourier encoding terms and, generally, coil sensitivity values, f is the image vector, and b is the vector of measured data from all coil receivers. When the number of elements in f, that is, the number of image pixels, is greater than the number of rows in E, that is, the number of acquired data points, the linear system in Eq. (1) becomes underdetermined and has an infinite number of possible solutions. The simplest way to isolate a single solution is to minimize l_2 norm of the residue, that is, to solve the following problem:

$$f = \arg\min \ \|Ef - b\|_2^2.$$

If E incorporates coil sensitivity values, then such minimization corresponds to the simplest formulation of parallel imaging (SENSE) approach [1]. However, decreased acquisition time and noise amplification (g-factor) lead to increased noise level in the reconstructed imaging.

Tikhonov Regularization. One of the ways to solve this problem is to apply Tikhonov regularization [11,12] or the regularization by l_2 norm:

$$f = \arg\min\left(\|Ef - b\|_2^2 + \lambda\|f\|_2^2\right).$$

where λ is the regularizing parameter that provides a balance between the level of the noise in the reconstructed image and the level of residual artifacts.

Compressed Sensing. Recently, a novel mathematical theory has been developed [6] that states that *sparse* images (i.e., images with a relatively small number of pixels containing relevant information) can be accurately reconstructed from undersampled datasets, provided the encoding matrix E satisfies certain conditions. Ideally, the sparsity of an image is measured by its l_0 norm that counts the number of non-zero pixels. Therefore, if we know in advance that the underlying image is expected to be sparse (as is the case, for example, in MR angiography), then the image may be obtained as

$$f = \arg\min\left(\|Ef - b\|_2^2 + \lambda\|f\|_0\right).$$

The problem with this formulation is that while it allows for an accurate reconstruction of sparse images, the minimization possesses combinatorial complexity, so its practical implementation is infeasible. However, the compressed sensing theory proves that, under certain conditions, the solution of l_0 minimization problem is equivalent to the solution of l_1 minimization problem, i.e. we can solve the following problem:

$$f = \arg\min\left(\|Ef - b\|_2^2 + \lambda\|f\|_1\right).$$

There are a number of computationally efficient ways to implement l_1 minimization in practice, which made compressed sensing ideas attractive to accelerated MR imaging [7, 8].

In compressed sensing, admissible acceleration factors are analytically related to the sparsity level of the underlying signal. Higher level of undersampling leads to artifacts in the reconstructed images. This often poses a problem in rapid imaging, since even intrinsically sparse angiographic images may not possess the level of sparsity necessary to support the high acceleration factors desirable in some applications. However, image sparsity can be enhanced either by an application of a sparsifying transform such as an image gradient or a wavelet transform, or by subtracting a prior image estimate [13], or by both. Moreover, several regularizing terms may be used to provide a better reconstruction. Therefore, in its most general formulation compressed sensing solves the following minimization problem:

$$f = \arg\min\left(\|Ef - b\|_2^2 + \sum \lambda_k \left\|\Psi_k(f - f_{0,k})\right\|_1\right).$$

Here, Ψ_k are sparsifying transforms and $f_{0,k}$ are corresponding prior image estimates that may be obtained in a number of ways, for example, from a prior scan, or from more densely sampled low frequencies, or from temporally averaging a time series.

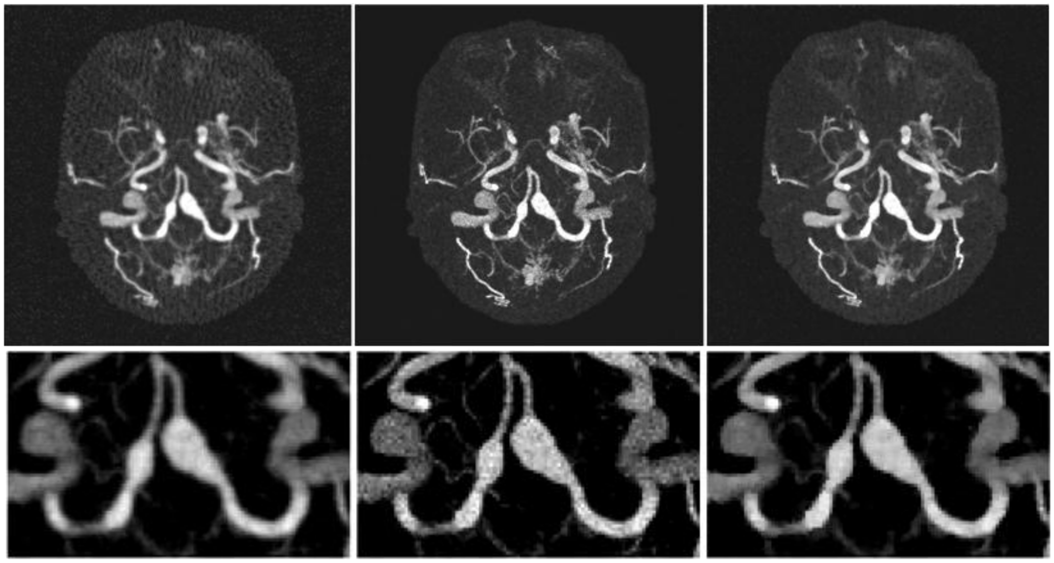

Figure 1. Reconstruction of an image from a radially undersampled dataset (acceleration factor 4) using l_2 regularization (left), l_1 regularization of the image itself (center), and l_1 regularization of the image gradient (right).

Images in Fig. 1 compare the effects of different ways to regularize reconstruction and provide an illustration to the fact that l_2 norm is optimal from the point of view of noise properties but does not eliminate residual artifacts; l_1 norm helps restore image sharpness but is not noise optimal; while l_1 norm of the image gradient provides a tradeoff between these two cases.

We will discuss both theoretical requirements of compressed sensing and some aspects of its practical implementation.

HYPR

The HighlY constract backPRojection (HYPR) method belongs to another family of constrained reconstruction algorithms, which use a multiplicative constraint by a prior image. HYPR reconstruction is usually applied to serial imaging, such as time-resolved imaging or diffusion tensor imaging. A HYPR image is obtained as

$$H = W \cdot C$$

where C is the prior image estimate and W is a weighting image. The prior image estimate, C, also called the composite image, is usually obtained from averaging all or a subset of the data collected during the exam. The quality of the composite image largely determines spatial resolution and SNR of the individual HYPR frames. The main distinction between different algorithms in the HYPR family lies in the way the weighting images are formed. The original HYPR algorithm [9] and its modification [14] use unfiltered backprojection, and therefore are tailored specifically to radial acquisition. The subsequently developed HYPR LR algorithm [10] relies on k-space filtering to form the weighting images and can be applicable to any sampling trajectory. Another advantage of HYPR LR is that it reduces signal cross-talk between spatially adjacent objects with different time courses, such as, for example, an artery and a vein. This

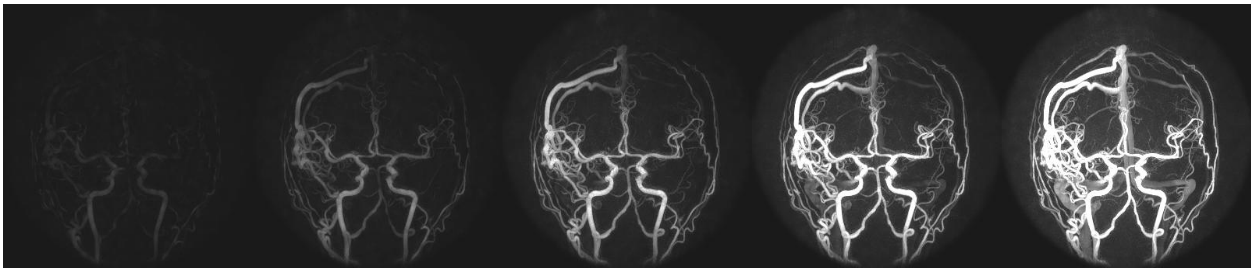

Figure 2. HYPR Flow reconstructed maximum intensity projections of 3D time frames showing contrast arrival in an AVM patient.

property allows for the use of composite images collected over a longer period of time and, thus, having higher SNR, which is then transferred to individual HYPR frames. Another possibility is to acquire a composite image in a separate scan, as was done in the HYPR Flow technique [15]. Images in Fig. 2 illustrate contrast arrival in an AVM patient using the HYPR Flow technique.

Both HYPR and HYPR LR algorithms are approximate image reconstruction techniques. We will discuss the dependence of the reconstruction error and performance of the algorithms on image sparsity and spatio-temporal correlation of the images in the series. We will also discuss several iterative HYPR techniques [16-18] that were developed with the aim of improving the accuracy of the reconstruction.

References

1. Pruessmann KP, Weiger M, Scheidegger MB, Boesiger P. SENSE: sensitivity encoding for fast MRI. Magn Reson Med 1999; 42(5):952-962.
2. Sodickson DK, Manning WJ. Simultaneous acquisition of spatial harmonics (SMASH): fast imaging with radiofrequency coil arrays. Magn Reson Med 1997; 38(4):591-603.
3. Griswold MA, Jakob PM, Heidemann RM, Nittka M, Jellus V, Wang J, Kiefer B, Haase A, Generalized autocalibrating partially parallel acquisitions (GRAPPA), Magn Reson Med 2002; 47(6): 1202-1210.
4. Madore B, Glover GH, Pelc NJ, Unaliasing by Fourier-encoding the overlaps using the temporal dimension (UNFOLD), applied to cardiac imaging and fMRI, Magm Reson Med 1999; 42(5): 813-828.
5. Tsao J, Boesiger P, Pruessmann KP, *k-t* BLAST and *k-t* SENSE: dynamic MRI with high frame rate exploiting spatiotemporal correlations, Magn Reson Med 2003; 50: 1031-1042.
6. Candès EJ, Romberg J. Sparsity and incoherence in compressive sampling. Inverse Problems 23:969-985.
7. Block KT, Uecker M, Frahm J. Undersampled radial MRI with multiple coils. Iterative image reconstruction using a total variation constraint. Magn Reson Med 2007; 57(6):1086-1098.
8. Lustig M, Donoho D, Pauly JM. Sparse MRI: The application of compressed sensing for rapid MR imaging. Magn Reson Med 2007; 58(6):1182-1195.
9. Mistretta CA, Wieben O, Velikina JV, Block WF, Perry J, Wu Y, Johnson KM, Wu Y, Highly constrained backprojection for time-resolved MRI, Magn Reson Med 2006; 55: 30-40.
10. Johnson KM, Velikina JV, Wu Y, Kecskemeti S, Wieben O, Mistretta CA, Improved waveform fidelity using local HYPR reconstruction (HYPR LR), Magn Reson Med 2008; 59: 456-62.
11. Tikhonov AN, Arsenin YV. Solution of Ill-posed Problems: Winston and Sons; 1977.
12. Lin FH, Kwong KK, Belliveau JW, Wald LL. Parallel imaging reconstruction using automatic regularization. Magn Reson Med 2004; 51(3):559-567.
13. Samsonov AA, Jung Y, Alexander AL, Block WF, Field AS. MRI compressed sensing via sparsifying images. In: Proc. of ISMRM, 2008, Toronto. p 342.

14. Huang Y, Gurr D, Wright GA, Three-dimensional reconstruction of limited-view projections for contrast-enhanced magnetic resonance angiography at high temporal and spatial resolution, Magn Reson Med 2006; 55(1): 68-74.
15. Velikina JV, Johnson KM, Wu Y, Samsonov AA, Turski P, Mistretta CA, PC HYPR Flow: a Technique for Rapid Imaging of Contrast Dynamics, J Magn Res Imag, 2010; 31(2):447-456.
16. 16. O'Halloran RL, Wen Z, Holmes JH, Fain SB, Iterative projection reconstruction of time-resolved images using highly-constrained back-projection (HYPR), Magn Reson Med 2008; 59: 132-139.
17. Griswold M, Barkasukas K, Blaimer M, Moriguchi H, Sunshine J, Duerk J, More optimal HYPR reconstructions using a combination of HYPR and conjugate-gradient minimization, in Proc. of 18th MRA Workshop, Basel, Switzerland, 2006.
18. Samsonov AA, Wieben O, Block WF. HYPRIT: Generalized HYPR Reconstruction by Iterative Estimation. *ISMRM Workshop on Non-Cartesian Imaging,* Sedona, AZ, February 2007.