A Unified Framework for SNR Comparisons of Four Array Image Combination Methods

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Purpose

This tutorial applies a shared mathematical framework to replace the widely varying notations seen in the literature allowing the comparison of four array image combination methods. The resulting signal-to-noise ratios (SNRs) are derived.

Definitions

Variable definitions are in the table; the corresponding array dimensions are those needed to calculate the value of a *single pixel* in the composite image. Noise variables are described by their statistical moments; expectation is denoted by $E[\cdot]$. The receiver noise covariance matrix is $\Psi_\eta = E[\eta \cdot \eta^H]$, where the tensor product produces a square matrix from vectors. The superscript H indicates conjugate (Hermitian) transpose. The combined estimated image pixel value, $\hat{\rho}$, results from a linear combination of the corresponding values from the single coils. In contrast to most of the literature we take the Hermitian transpose of the weights as in Ref. (1), without loss of generality. Noise in the final image is $\varepsilon = W^H \eta$ whose variance is then $\sigma_\varepsilon^2 = E[\varepsilon \cdot \varepsilon^H] = E[W^H \eta \cdot \eta^H W] = W^H \Psi_\eta W$, consistently with Ref. (2).

Variable description	symbol	set
Number of coils	N	$\in \mathbb{N}$
Coil sensitivities	s	$\in \mathbb{C}_{N\times 1}$
Single-coil pixel data	d	$\in \mathbb{C}_{N\times 1}$
Noise in coil data	η	$\in \mathbb{C}_{N\times 1}$
Noise in final image	ε	$\in \mathbb{C}$
Coil data weights	W	$\in \mathbb{C}_{N\times 1}$
Actual image value	ρ	$\in \mathbb{C}$
$\hat{\rho} = W^H d = \rho + \varepsilon \qquad d = \rho \cdot s + \eta = \hat{\rho} \cdot s$		

Weighting functions and associated SNR

1. **Sensitivity-based**: if noise statistics are unknown then it makes sense to weigh the data by the relative magnitude of each coil's sensitivity, $W = s(s^H s)^{-\frac{1}{2}}$, from which $SNR = \hat{\rho}\sigma_\varepsilon^{-1} = W^H d (W^H \Psi_\eta W)^{-\frac{1}{2}} = s^H d (s^H \Psi_\eta s)^{-\frac{1}{2}}$ (Eq. [8] of Ref. (2)).

1a. Root-sum-of-squares (RSS): assuming the data itself is a reasonable approximation of the coil sensitivity ($s \approx d$) the above reduce, respectively, to $\hat{\rho} = \sqrt{d^H d}$ and $SNR = d^H d (d^H \Psi_\eta d)^{-\frac{1}{2}}$, which is Eq. [4] of Ref. (3). If the data is pre-whitened (e.g., using "eigencoil" hardware (3)) the SNR is the image itself. RSS is convenient but does suffer from bias ($E[\hat{\rho}] \neq \rho$) (4).

2. **Spatial matched filter**: this SNR-optimal combination is also known as maximum-ratio or "Roemer" reconstruction (5). Noise covariance is included in the weighting along with sensitivity, $W = \Psi_\eta^{-1} s$, resulting in noise variance in the composite image $\sigma_\varepsilon^2 = W^H \Psi_\eta W = s^H \Psi_\eta^{-1} \Psi_\eta \Psi_\eta^{-1} s = s^H \Psi_\eta^{-1} s$ and $SNR = s^H \Psi_\eta^{-1} d (s^H \Psi_\eta^{-1} s)^{-\frac{1}{2}}$ (Eq. [32] in Ref. (5)).

2a. Weighted root sum-of-squares (wRSS): approximating coil sensitivity with the data itself ($s \approx d$) results in $\hat{\rho} = d^H \Psi_\eta^{-1} d$ and $SNR = \sqrt{d^H \Psi_\eta^{-1} d}$ (Eq. [24] of Ref. (5)), which is a convenient SNR estimate for the matched filter combination (6).

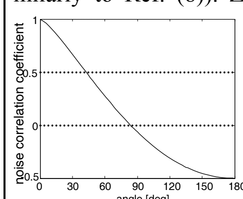
Discussion and SNR comparisons

Noise covariance *always* influences SNR, regardless of whether it is used in W or not, and appears differently in the two classes of image combination. Nevertheless, the SNRs are equal when Ψ_η is identity as well as when SNR $\rightarrow \infty$ (4). Within the two classes the SNRs are essentially identical since typically $s \approx d \cdot d_0^{-1}$, where d_0 is a reference (body coil) pixel value that cancels out leaving the same SNR as with $s \approx d$. Including sensitivities does allow special forms of W that give "uniform" sensitivity images (i.e., weighted by d_0) (2,5), potentially with less bias.

Comparisons of SNR in simulated (Fig. 1) and measured (Fig. 2) data are shown. The maximal SNR difference between the two classes depends on N: ~10% for a two-coil array ((5) and Fig. 1) and greater for larger systems ((7) and Fig. 2).

Summary

Relationships between four methods of array image combination have been expressed using a shared mathematical framework. The role of noise covariance in achieving optimal SNR is highlighted using examples.



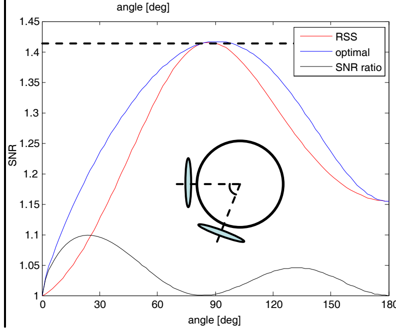
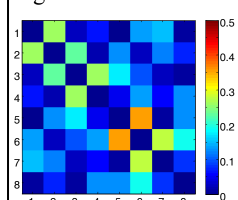


Figure 1: simulated noise correlation (normalized covariance) and SNRs at the centre of a spherical phantom as a function of the angle separating two coils (similarly to Ref. (8)). Zero correlation is not the only situation in which the SNRs are equal. Dashed line corresponds to $\sqrt{2}$.



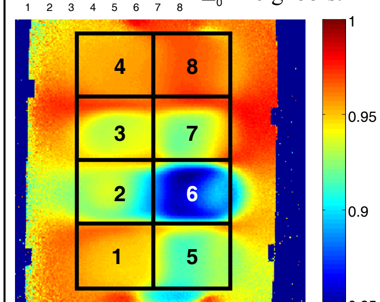


Figure 2: noise correlation (diagonal set to zero) and SNR map of RSS combination relative to matched filter. The grid indicates coil layout and numbering. Noise covariance is critical for optimal SNR when elements are strongly coupled (corr > 0.3), e.g., element 6 and its neighbors.

References

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