Validity of the noncentral chi model in multiple-coil systems with noise correlations

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Noise in Multiple-Coil Systems(2)

Noise is one of the main sources of quality deterioration in Magnetic Resonance (MR) data. It is usually modeled attending to the scanner coil architecture. The complex spatial MR data is typically assumed to be a complex Gaussian process, where the real and imaginary parts of the original signal are corrupted with uncorrelated Gaussian noise with zero mean and equal variance σ_n^2. Thus, the magnitude signal in single-coil systems is the Rician distributed envelope of the complex signal. In multiple-coil MR acquisition systems, the process is repeated for each receiving coil. As a consequence, the noise in the complex signal in the **x**-space for coil l-th (l=1,..,L) will also be Gaussian. If the composite magnitude signal (CMS) is obtained using sum-of-squares (SoS) it can be modeled as a noncentral chi (nc-χ) distribution [Const97]. However, the CMS will only follow a nc-χ distribution if the variance of noise is the same for all coils, and no correlation exists between them. Although it is well known that in phased array systems noise correlations do exist [Hayes90], the effect of noise correlations is usually left aside due to their minimal effect and practical considerations. In [Const97] it is stated that the effect of such correlations is minimal over humans and phantoms. However, these correlations may affect the final statistical model and the effective level of noise in the image.

$$\mathbf{\Sigma}^2=\sigma_n^2\begin{pmatrix}1 & \rho^2 & \cdots & \rho^2\\ \rho^2 & 1 & \cdots & \rho^2\\ \vdots & \vdots & \ddots & \vdots\\ \rho^2 & \rho^2 & \cdots & 1\end{pmatrix} \quad \textbf{(1)}$$

Let us assume the simple covariance matrix between coils in eq. (1). Following a similar reasoning to the one in [Aja10] it is evident that the CMS after SoS reconstruction will not strictly follow a nc-χ. However, our hypothesis is that for low values of the coefficient of correlation (ρ^2) the nc-χ distribution can be used as a good approximation of the actual one. The correlation between coils will translate in a decrease of the number of Degrees of Freedom (DoF) of the distribution. Thus, the resulting distribution will show a (reduced) *effective number of coils* and an (increased) *effective variance of noise* that can be calculated using the method of the moments, see eq. (2), with $\mathbf{A}(\mathbf{x})=[A_1,\cdots,A_L]^T$ the signal in each coil and $\|.\|^2_F$ the Frobenius norm.

$$L_{\text{eff}}=\frac{|\mathbf{A}|^2\ \text{tr}\left(\mathbf{\Sigma}^2\right)+\left(\text{tr}\left(\mathbf{\Sigma}^2\right)\right)^2}{\mathbf{A}^*\mathbf{\Sigma}^2\mathbf{A}+\|\mathbf{\Sigma}^2\|_F^2}$$

$$\sigma^2_{\text{eff}}=\frac{\text{tr}\left(\mathbf{\Sigma}^2\right)}{L_{\text{eff}}} \quad \textbf{(2)}$$

$$L_{\text{eff}}=\frac{L}{1+(L-1)\rho^4} \quad \textbf{(3)}$$

$$\sigma^2_{\text{eff}}=\sigma_n^2\left(1+(L-1)\rho^4\right)$$

In the background of the image, where the signal is zero (and the distribution is therefore a central chi, c-χ), these equations simplify to eq. (3), where L is the number of coils of the system. As a result, the effective number of coils is expected to be reduced due to the correlation between coils, while the effective noise variance is expected to increase. The existence of this effective values was experimentally observed in [Thumb07].

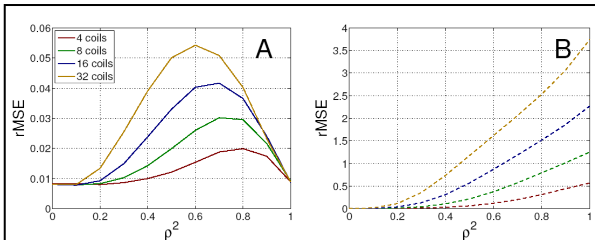

Fig. 1: Relative errors in the PDF for the non-central Chi approximation, as a function of the correlation coefficient. A) Using effective parameters. B) Using the original parameters.

Correlation and the noncentral chi model : experimental results

Here we present some experiments supporting the initial hypothesis. **First** the SoS of 128x128 Gaussian correlated synthetic images are considered: 0 mean, σ_n^2=1, ρ^2 ranging in [0-1] and different coil architecture (4 to 32 coils). The analysis in terms of ρ^2 is used to characterize whether or not a nc-χ model can be accurately fitted in each particular situation. To that end, the (relative) Mean Squared Error (rMSE) in eq. (4) is measured; $g_\chi(x)$ is the real distribution of the CMS obtained by SoS and $\tilde{g}_\chi(x)$ is the distribution of an *equivalent* nc-χ. 100 different realizations are considered and the average of the error is plotted in Fig. 1 as a function of the ρ^2 for different coil number. Fig. 1-A shows the rMSE when effective parameters are considered, and Fig. 1-B shows the rMSE using the original parameters. It is clear that the original parameters cannot be used to model the actual distribution, since the error committed will be huge. Using the effective parameters, the error when a nc-χ is considered is significantly reduced, for low correlation, it is almost null. The error grows with ρ^2, and it is more significant for larger coil number. As a final comment, note that when the correlation is very high, it is similar to have the same image, so the error decreases; it will be equivalent to having a single coil system. The effective number of coils estimated for each coefficient of correlation is shown in Fig. 2. As the correlation grows, all the values tend to one. Note that those configurations with a greater number of are the ones with the greater slope (see the relative values in Fig. 2-B). That means that they are much more affected by correlations. For instance, a 32-coil configuration with ρ^2=0.2 will be equivalent to a 18-coil configuration. This result gives us an idea of the importance of the correlations in the performance of the scanner. Less correlated coils will mean higher effective number of coils, and therefore a smaller value of effective variance of noise.

$$\text{rMSE}=\frac{\int|g_\chi(x)-\widetilde{g_\chi}(x)|^2dx}{\int|\tilde{g_\chi}(x)|^2dx} \quad \textbf{(4)}$$

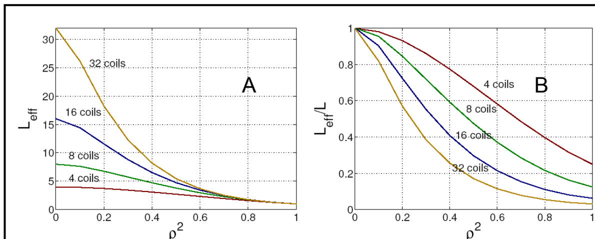

Fig. 2: Effective number of coils as a function of the coefficient of correlation. A) Absolute value. B) Relative value.

A **second** experiment is carried out using real data: 100 repetitions of the same slice of a phantom, see Fig. 3, scanned in an 8-channel head coil on a GE Signa 1.5T EXCITE 12m4 scanner with FGRE Pulse Sequence to generate low SNR, Matrix size= 128x128, TR/TE=8.6/3.38 ms, FOV 21x21cm, slice thickness = 1mm. Noise variance is initially estimated using the variance of the real part of every coil of every sample, where the noise is known to be additive, with an estimated value σ_n = 0.0430. The effective values are calculated fitting a c-χ to the histogram of the background data of the CMS. Final results are:

L=8 L_{eff}=7.3073 σ_n = 0.0430 σ_{eff} = 0.0447

As expected, when real data is considered, the number of coils (parameter L of the c-χ distribution) is not the original value, but a reduced one. On the other hand, the variance of noise, when compared with the estimated in each coil, shows a greater value, due to the correlation between coils. Finally, the correlation coefficient can also be estimated from the effective values using eq. (3): ρ^2=0.1112.

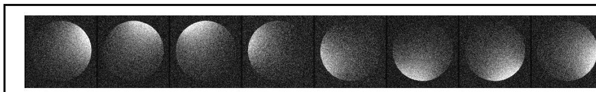

Fig. 3. Slice of an 8-coil 2D acquisition of a doped ball phantom.

Conclusions: Due to the correlation between coils in multiple-coil systems, the distribution of the CMS is not strictly a nc-χ. However, it can be modeled as such if the coefficient of correlation is small enough and effective values are considered. The main consequence of the correlation between coils is a reduction in the effective number of coils in the system. In systems with a large number of coils (32, 64…) this reduction could be quite significant. In addition, the reduction of the effective number of coils implies an increase in the effective variance of noise. To sum up, to increase the effectiveness of the scanner, an increase of the number of coils should go together with a reduction of the correlation between them.

[Aja10] Aja-Fernández, *et* al.Statistical noise analysis in GRAPPA using a parametrized noncentral chi approximation model, *MRM*, *in press.*
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[Hayes90] C. Hayes, P. B. Roemer, Noise correlations in data simultaneously acquired from multiple surface coil arrays, *MRM*. 16:181–191 (1990).
[Thumb07] Thumberg *et al.* Noise distribution in SENSE- and GRAPPA-reconstructed images: a comp. simulation study, *MRI* 25:1089 (2007)