A theoretical analysis of the Morphology Enabled Dipole Inversion (MEDI) method: using anatomical information to improve the calculation of susceptibility

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INTRODUCTION There are several empirical approaches to determining tissue magnetic susceptibility property or quantitative susceptibility mapping(QSM) [1-2]. The purpose of this work is to establish a firm theoretic base for QSM. Fundamentally, we need to solve an ill-posed inverse problem from the magnetic field measured in MRI phase data to the tissue susceptibility distribution. The convolution kernel in the forward problem from susceptibility distribution to magnetic field is a dipole kernel that is zero at the 54.7^0 magic angle in its Fourier expression. Thus, many susceptibility distributions can generate the same field. However, the dipole zero surface only causes slight undersampling of the susceptibility in the Fourier domain, and the susceptibility can be uniquely determined using a little addtional information, such as structural information from the magntiude images. We term this approach as morphology enabled dipole inversion (MEDI) [3-4].
Here we provide a detailed analytical and numerical analysis to demonstrate that this MEDI approah provides a unique and accurate solution under ideal conditions, and the error of the reconstruction is tightly bounded by the error in the gradient echo image in the presence of Gaussian noise.

MATERIALS AND METHODS The inverse problem from measured local field δ to the unknown susceptibility χ is formulated as a constrained minimization problem:

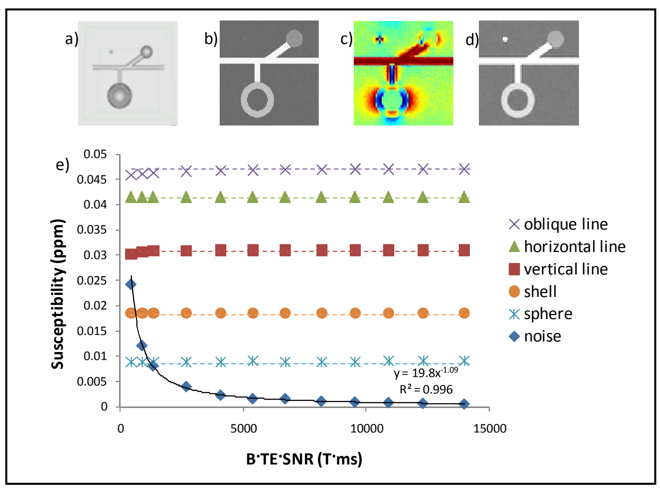
$$\min_{\chi}\|M\nabla\chi\|_1 \; s.t. \; \|W(\delta - F_D\chi)\|_2 \le \varepsilon, \quad (1)$$

where M is a binary weighting diagonal matrix generated from the gradient of the magnitude image, W is a weighting matrix compensating for the non-uniform phase noise, F_D is $F_D = F^{-1}DF$ with F being the 3D Fourier transform and D being a diagonal matrix whose diagonal elements are $1/3 - k_z^2/k^2$, and ε is chosen to correspond with the expected noise level. The matrix M is a modified identity matrix with zeros at locations corresponding to interface voxels. If we denote $\chi' = \nabla\chi$, then the problem can be reformulated in terms of the susceptibility gradient:

$$\min_{\chi'}\|M\chi'\|_1 \; s.t. \; \|W(\delta - F_D\nabla^+\chi')\|_2 \le \varepsilon, \mathbf{curl}\chi' = 0, \quad (2)$$

where the constraint $\mathbf{curl}\chi' = 0$ is added to ensure that χ' represents a gradient field. The Helmholtz theorem states $\forall v', \mathbf{curl}v' = 0 \Leftrightarrow v' \in \mathrm{CS}(\nabla)$ (CS: column space). The Moore-Penrose pseudo-inverse of ∇ is denoted ∇^+ and it is determined at all points in the Fourier expression, except at the origin where it is set to zero which exclusively affects a constant term in the susceptibility distribution.
It is assumed that 1) the true susceptibility distribution χ_0 is approximately piece-wise constant or equivalently $\chi_0{}'$ is approximately sparse, and 2) the locations of most of the non-zero elements in $\chi_0{}'$ are known from the magnitude image and the corresponding diagonal elements in M are assigned 0. These two assumptions may be written as $\|M\chi_0{}'\|_1/\|\chi_0{}'\|_1 \ll 1$. The observation $\delta = F_D\nabla^+\chi_0{}' + e$ is contaminated by noise e and the weighted ℓ_2 norm of the error is $\|We\|_2 = \varepsilon$. The solution to Eq. 1 and Eq. 2 are respectively denoted as $\hat{\chi}$ and $\widehat{\chi'}$.

RESULTS
Lemma 1: The reconstructed gradient error $\zeta' = \widehat{\chi'} - \chi_0'$ satisfies $\|WF_D\nabla^+\zeta'\|_2 \le 2\varepsilon$ and $\|M\zeta'\|_1 \le 2\|M\chi_0{}'\|_1$. This Lemma can be proven by the triangular inequality.
Theorem 1: If the gradients in the interface voxels can be determined from the measured local field, then the ℓ_2 norm of the error of the reconstructed susceptibility $\zeta = \hat{\chi} - \chi_0$ is linearly bounded by the input error, which consists of the input model error defined as $\|M\chi_0{}'\|_1$ and the input measurement error defined as ε. If additionally there is no input error, the solution is unique and accurate. This theorem can be proved by applying Lemma 1 and establishing the following bond, $\|\zeta'\|_2 \le \|(I-M)\zeta'\|_2 + \|M\zeta'\|_2 \le \frac{2}{r_1}(\varepsilon + R\|M\chi_0{}'\|_1) + 2\|M\chi_0{}'\|_1$, and $\|\zeta\|_2 \le \|\zeta'\|_2/r_2$, where $r_1 = \min_{v'\in \mathrm{CS}(\nabla),(I-M)v'\neq 0} \|(WF_D\nabla^+)(I-M)v'\|_2/\|(I-M)v'\|_2 > 0, r_2 = \min_{\nabla v\neq 0, v\neq 0} \|\nabla v\|_2/\|v\|_2 > 0$, $R = \max_{v'\in \mathrm{CS}(\nabla),Mv'\neq 0} \|(WF_D\nabla^+)Mv'\|_2/\|Mv'\|_2$ are finite numbers. In particular, r_1 is sufficiently larger than zero as F_D has only limited zeroes.
Sensitivity is limited by noise such that the susceptibility value should be above the noise level (Rose model). Noise propagation in MEDI reconstruction is determined by numerical analysis with results shown in Figure. The MRI of a physical object (a) consists of a magnitude image (b) reflecting materials' T1, T2 and proton density, and a phase image (c) resulting from materials' varying susceptibility. Combining the structural information from the magnitude image, the field-to-source ill-posed inverse problem is solved (d). We verified r_1, r_2 and R in this simulation, where $r_1=1.5\times10^{-3}$, $r_2=0.10$, $R=0.53$. The estimated mean susceptibilities in various objects in (a) are plotted with the dashed lines indicating the true susceptibilities in the simulation (e). Standard deviation of noise measured from the background is fitted to a power function. The mean susceptibilities are accurate and independent of imaging conditions, while noise is approximately inversely proportional to the product of B, TE and SNR. The leftmost point corresponded to B = 1.5T, TE = 10ms, SNR = 30. The rightmost point corresponded to B = 7.0T, TE = 40ms, SNR = 50.



CONCLUSION The mathematical analysis here establishes the existence of a sufficiently accurate and unique solution to the field to susceptibility source inverse problem. This existence proof provides a fundamental justification for developing the MEDI approach that makes use of the structural information in the magnitude images. The magnetic field to susceptibility source inverse problem in MRI may be regarded as a fortunate inverse problem: the structural information is readily available from the magnitude images of MRI. Slight undersampling of susceptibility in the field/phase images causes some streaking artifacts in QSM, so a little information from the magnitude is sufficient for selecting a streaking-free solution. The information in QSM is primarily coming from the field or the phase images.

REFERENCES [1] Kressler et al. IEEE TMI:29:273; [2] Shmueli et al. MRM:62:1510; [3] de Rochefort et al. MRM:63:194; [4] Liu et al. ISMRM Proc:2010:4996.