Gaussian dephasing due to finite gradients in q-space imaging

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Introduction

Diffusion imaging is a powerful tool for probing the microstructure of biological tissue. With q-space imaging (QSI), the probability density function (PDF) of the diffusion process (or more exactly, the averaged propagator) can be obtained in terms of a Fourier transform of the MR-signal for different diffusion encodings (1,2). This allows one to detect e.g. deviations from a Gaussian PDF in case of restricted diffusion. However, a fundamental assumption in QSI is the so-called short gradient approximation (SGA) which can not be satisfied on clinical scanners.

We present a new formalism for QSI that is also valid for long duration diffusion encoding gradients. It starts from the basic equations for Brownian motion and magnetic field gradient encoding. The general solution of the resulting Fokker-Planck equation shows that the SGA is oversimplifying the situation. It turns out that the effect of finite gradients is to make the propagator more Gaussian.

Material and methods

To simplify the mathematics, we consider the one dimensional case. For Brownian motion (treated as a Wiener process) along the x -direction and diffusion encoding gradient with waveform $g(t)$, we can write the following system of stochastic differential equations:

$$\frac{dx}{dt} = \sqrt{2D}\xi(t) \qquad \text{and} \qquad \frac{d\varphi}{dt} = \gamma g(t)x(t) \qquad [1]$$

with $x(t)$ the time dependent position of the diffusing spins, D the diffusion coefficient, $\xi(t)$ the white noise (i.e. delta correlated) driving function, $\varphi(t)$ the phase shift and γ the gyromagnetic ratio. The equivalent Fokker-Planck equation (FPE) is then:

$$\frac{\partial p}{\partial t} = -\gamma g(t)x\frac{\partial p}{\partial \varphi} + D\frac{\partial^2 p}{\partial x^2} \qquad [2]$$

with $p(x,\varphi,t)$ the joint PDF of position and phase at time t . In the absence of gradient encoding, eq.[2] reduces to the standard diffusion equation. The diffusion propagator can always be expressed in terms of its spectral components:

$$p_D(x,t) = \frac{1}{2\pi}\int P_D(k,t=0)\,e^{-Dk^2t}e^{ikx}dk \qquad [3]$$

Fourier transforming the FPE eq.[2] with respect to x and φ gives:

$$\frac{\partial P}{\partial t} = -\gamma g(t)k_\varphi\frac{\partial P}{\partial k_x} + Dk_x^2 P \qquad \text{with } P(k_x,k_\varphi,t) \qquad [4]$$

The general solution of eq.[4] can be found using the method of characteristics and written as:

$$P(k_x,k_\varphi,t) = P(k_x + k_\varphi[q(t)-q(t_0)],t_0).\exp\left\{-D\int_{t_0}^{t} d\lambda\left[k_x + k_\varphi\{q(t)-q(\lambda)\}\right]^2\right\} \qquad \text{with} \qquad q(t) = \gamma\int_0^t g(s)ds \qquad [5]$$

The MR-signal is then obtained as $S(t) = P(0,1,t)$. This formalism is applied for a pulsed gradient spin echo sequence with rectangular diffusion encoding gradients with strength g , pulse duration δ and a time delay Δ between the onset of the two pulses.

Results and discussion

For impulsive gradients, eq.[5] reduces to the SGA: $S(q) = P(q,0).\exp\left[-Dq^2\Delta\right] = P(q,\Delta)$ with $q = \gamma g\delta$. This expresses the well-known Fourier relationship between MR-signal and averaged propagator after a time delay Δ . In general, the joint PDF $p(x,\varphi,t)$ can be written as:

$$p(x,\varphi,t) = p(x + \frac{\Gamma_1}{\Gamma_2}\varphi, t - \frac{\Gamma_1^2}{\Gamma_2}).\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\varphi^2}{2\sigma^2}} \qquad \text{with} \qquad \sigma = \sqrt{2D\Gamma_2} \qquad \text{and} \qquad \Gamma_n = \int_0^t q^n(s)ds \qquad [6]$$

and reveals the inherent Gaussian phase distribution due to the white noise. The averaged propagator can then be written in terms of a convolution:

$$\overline{p}(R) = \overline{p}(R,t-\tau)\otimes\exp\left[-R^2/(4D\tau)\right]/\sqrt{4\pi D\tau} \qquad \text{with} \qquad \tau = \Delta^2/(\Delta - \delta/3) \qquad [7]$$

Fourier transformation with respect to the displacement R gives: $S(q) = P(q,t-\tau)\exp(-q^2D\tau) = P(q,t)$ with $t = \Delta + \delta$. This shows that it is difficult to interpret diffusion encoding in terms of a diffusion time in case of finite gradients: a delay of the propagator compensates attenuation due to the gradients. Based on eq.[5] combined with $S(t) = P(0,1,t)$, a possible interpretation can be expressed as $S(q) = \overline{P}_D(q,\Delta)\exp(-q^2D\delta/3)$. According to this interpretation, the effect of the finite gradients is to make the attenuation more Gaussian. It should be noted that this Gaussian character is not taken into account in multiple narrow pulse and matrix formalisms (2).

In conclusion, our formalism reveals the inherent Gaussian dephasing associated with diffusion encoding. This Gaussian character will always be present in descriptions based on the Wiener process and thus also if based on the diffusion equation and Bloch-Torrey equation.

References

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