

# Apparent Kurtosis in the Motional Narrowing Regime: Analytic Results for Closed Domains

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## Introduction

Diffusion kurtosis tensor imaging is a method to quantify non-Gaussian diffusion [1]. Using elongated diffusion gradients, the apparent kurtosis  $K_{app}$ , but not the actual kurtosis  $K$  is measured. Since the relation between  $K_{app}$ , the shape of the temporal diffusion gradient profile and the geometry is mostly elusive to date, we investigated this relation for closed geometries and diffusion times that are typical for current in vivo diffusion imaging protocols.

## Methods

We use the multiple correlation function (MCF) approach [2,3,4] in combination with the partial integration approach described in [5]. The MCF approach uses the eigenfunctions  $u_m(\mathbf{r})$  and dimensionless eigenvalues  $\lambda_m$  of the Laplace operator. We assume Neumann boundary conditions. The matrices  $B_{n,m} = \int_{\Omega} u_n^*(\mathbf{r}) u_m(\mathbf{r}) B(\mathbf{r}) d\mathbf{r}$  are derived from the eigenfunctions and the spatial profile of the applied magnetic field  $B(\mathbf{r})$ . Here, we use spatially linear gradients. We consider 4 effective temporal gradient profiles.  $f_{st}(t, \delta) = -1$  for  $0 < t < \delta$  and  $1$  for  $t \geq \delta$  and  $0$  otherwise,  $f_{delta}(t) = \lim_{\delta \rightarrow 0} f_{st}(t, \delta)$ ,  $f_{cos}(t) = \cos(2\pi t)$ , and  $f_{sin}(t) = \sin(2\pi t)$ . We define  $\zeta_{n_1, n_2, n_3} = \sum_{m_1, m_2, m_3=1} B_{0, m_1} B_{m_1, m_2} B_{m_2, m_3} B_{m_3, 0} \lambda_{m_1}^{n_1} \lambda_{m_2}^{n_2} \lambda_{m_3}^{n_3}$  in analogy to  $\zeta_{n_1} = \sum_{m_1=1} B_{0, m_1} B_{m_1, 0} \lambda_{m_1}^{n_1}$  [4] (see Table 2). The long time limit is defined such that terms containing  $\exp(-\lambda DT/L^2)$  are negligible ( $L$  is e.g. the radius of the sphere and  $D$  is the free diffusion constant).

## Results

In the long time limit the influence of the effective temporal gradient profile and the eigensystem decouple; we find the following expression

$$K_{app} = \frac{\mathbb{E}\{\varphi^4\} - 3\mathbb{E}\{\varphi^2\}^2}{\mathbb{E}\{\varphi^2\}^2} \approx 6 \frac{I_3(\zeta_{-1,-1,-1} - \zeta_{-1}\zeta_{-2})p^{-3} - I_4(2\zeta_{-1,-2,-1} - \zeta_{-2}^2)p^{-4} - I_5(\zeta_{-1,-1,-2} - \zeta_{-1}\zeta_{-3})p^{-4} + O(p^{-5})}{\mathbb{E}\{\varphi^2\}^2} \quad (1)$$

where  $I_3$ ,  $I_4$ , and  $I_5$  are real valued numbers that depend solely on the effective temporal gradient profile (Table 1) and the  $\zeta$ -constants depend solely on the eigensystem and the spatial gradient profile.

Figures: Apparent kurtosis  $K_{app}$  (thick line) in dependency of normalized diffusion time  $p = DT/L^2$  computed using the MCF approach. The thin lines represents the motional narrowing regime approximation to order  $p^{-4}$  in the numerator of eq. (1) (continuous thin line), and to order  $p^{-3}$  (dotted thin line). In the motional narrowing regime, the approximation is in perfect agreement with  $K_{app}$ .

## Discussion

Eq. (1) is a large step forward in understanding the relation between  $K_{app}$  and eigensystem. It clearly separates the effect of temporal gradient profile and eigensystem and allows to determine the constants  $\zeta_{n_1, n_2, n_3}$ , which are solely dependent on the geometry of the confining domain and the spatial profile of the gradients. Unfortunately, we are not aware of a direct link between  $\zeta_{n_1, n_2, n_3}$  and general geometrical features (like the surface-to-volume ratio). Eq. (1) is valid for temporal gradient profiles that are not extremely curved, and which do not contain closely separated discontinuities. Although Eq. 1 requires the domain to be closed, it can serve as a starting point to investigate systems with permeable membranes and typical tissue parameters: For a cell of  $5 \mu\text{m}$  radius,  $100 \text{ ms}$  diffusion time and  $D = 1 \mu\text{m}^2/\text{ms}$ ,  $p$  is equal to  $4$  – meaning that Eq. (1) is valid for instance for  $f_{lin} = f_{st, \delta=1/2}$ .

**References** [1] Jensen et al. MRM 53, 1432 (2005) [2] Barzykin. J. Magn. Reson. 139, 342 (1999) [3] Axelrod et al. J. Chem. Phys. 114, 6878 (2001) [4] Grebenkov. Rev. Mod. Phys. 79, 1077 (2007) [5] Laun et al. ISMRM 2010, p.1581

Table 1		$f_{st, \delta < 1/2}$	$f_{st, \delta = 1/2} = f_{lin}$	$f_{sin}$	$f_{cos}$
$I_3$	$\int_0^1 f^4(t) dt$	$2\delta$	1	3/8	3/8
$I_4$	$(f_+^2 - f_-^2)^2/4$	1	1/2	0	1/2
$I_5$	$f_+^4 + f_-^4 - f_+^3 f_- - f_-^3 f_+$	4	6	0	2

Table 2	sphere
$\zeta_{-1}$	8/175
$\zeta_{-2}$	83/7875
$\zeta_{-3}$	2458 /1010625
$\zeta_{-1,-1,-1}$	1.8124e-4
$\zeta_{-1,-1,-2}$	4.1666e-5
$\zeta_{-1,-2,-1}$	1.5677e-5

