Scalable anti-aliasing image reconstruction in the presence of a quadratic "phase-scrambling" gradient using the fractional Fourier transform

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INTRODUCTION: Previous work has shown that scalable reconstructions can be obtained by pulsing a "phase-scrambling" quadratic gradient field prior to readout. Variable-FOV reconstructions have been achieved by means of the Frensel transform [1], [2] as well as the chirp-z transform [3]. This permits reconstruction of alias-free images from undersampled data, albeit with the final resolution limited by the number of acquired k-space points. In this work, we show that phase-scrambled MR data can be described as a mapping of the object known as a fractional Fourier transform (FrFT), a tool with a growing body of applications in optics and signal processing. We show the mathematical equivalence of the FrFT and Fresnel transform approaches. We then demonstrate scalable FrFT reconstructions with MR data acquired using a powerful, imaging-grade quadratic gradient insert coil applied before the readout.

THEORY: The Fresnel, fractional, and Fourier transforms are all subsets of a unitary integral

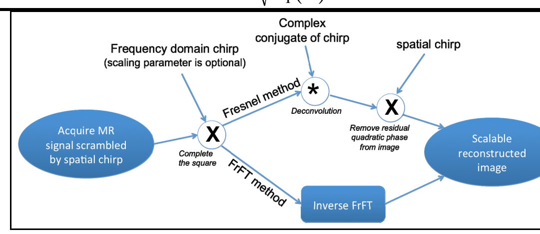
$$LCT_{a,b,c,d}\{f\}(u)=\sqrt{\frac{1}{ib}}\,e^{i\pi\left(\frac{d}{b}\right)u^2}\int e^{-i2\pi\left(\frac{1}{b}\right)uu'}e^{i\pi\left(\frac{a}{b}\right)u'^2}f(u')du',\ \ if\ b\neq 0 \qquad OR \qquad LCT_{a,b,c,d}\{f\}(u)=\sqrt{d}\,e^{i\pi cdu^2}f(d\cdot u),\ \ if\ b=0 \qquad (1)$$

operator known as the *linear canonical transform* (LCT), which is completely specified by the parameters a*,* *b*, *c*, and *d* (1). The LCT causes an affine transformation of the operand in time-frequency space. Its parameters are described by a 2×2 "*ABCD*" unit-determinant matrix with the fundamental property that the parameter matrix of a compound LCT is the product of the matrices of the component LCTs. The FrFT takes the form of a rotation matrix whose angle α is related to the *fractional order* a=$2\alpha/\pi$, repeating modulo 2π or 2, respectively. At α=0 the FrFT reduces to the identity operator, while at $\alpha=\pi/2$ it becomes the Fourier transform (eq. (2)). The Fresnel transform, which in optics describes wavefront propagation through free space, can be described as the convolution of the object (or light source) with a quadratic "chirp" integral kernel, shown in eq. (3).

Operation	LCT *ABCD* Matrix	Explicit formula
Fourier transform	$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$	$F\{f\}(u)=\int \exp(-i2\pi uu')f(u')du'$ (2)
Chirp convolution (Fresnel transform)	$\begin{bmatrix}1 & r\\ 0 & 1\end{bmatrix}$	$Fresnel\{f\}(u)=\sqrt{\frac{1}{ir}}\int \exp\left(\frac{i\pi(u-u')^2}{r}\right)f(u')du'=f(u)\otimes\sqrt{\frac{1}{ir}}\exp\left(\frac{i\pi u^2}{r}\right)$ (3)
Chirp multiplication	$\begin{bmatrix}1 & 0\\ -q & 1\end{bmatrix}$	$Q\{f\}(u)=f(u)\cdot\exp(-i\pi qu^2)$ (4)
Fractional Fourier transform*	$\begin{bmatrix}\cos(\alpha) & \sin(\alpha)\\ -\sin(\alpha) & \cos(\alpha)\end{bmatrix}$	$FrFT_\alpha\{f\}(u)=\sqrt{1-i\cot(\alpha)}\exp(i\pi\cot(\alpha)u^2)\int\exp(-i2\pi\csc(\alpha)uu')\exp(i\pi\cot(\alpha)u'^2)f(u')du'$ (5)

** We here present the FrFT as it most commonly appears in the optics literature: with an additional phase constant of* $\sqrt{\exp(i\alpha)}$ *outside the integral.*

The Fresnel transform and FrFT offer equivalent ways of reconstructing phase-scrambled data. Assume that an object $f(u)$ is imaged using a Cartesian sequence with a quadratic boxcar pulse of strength γH applied for τ sec prior to readout, giving a gradient moment $-\gamma H\tau$. Let *u* be a unitless spatial coordinate, $u=x\sqrt{N}/FOV$, with *N* voxels in the *FOV*. In the discrete case, *u* and *k* are both normalized to $k=[-N/2,\ N/2-1]/\sqrt{N}$. If the acquired signal is multiplied by the appropriate chirp function, the signal equation is recast in the form of a FrFT or a convolution with a Fresnel chirp kernel. In the Fresnel approach, image reconstruction proceeds via a deconvolution with the Fresnel kernel, realized either as convolution with the complex conjugate of the kernel [2] or via the inverse filtering method in the Fourier domain [1]. The image is then multiplied by the same chirp function, evaluated in spatial coordinates, to remove any residual quadratic phase. In the present work, we perform the same operations using the FrFT, exploiting the fact that LCT matrices can be decomposed into multiple forms of elementary LCT operations. For instance, the FrFT can be decomposed into three LCT matrices [4]: a chirp multiplication, a Fresnel transform, and a second chirp multiplication (eq. (6)).



$$\begin{bmatrix}\cos(\alpha) & \sin(\alpha)\\ -\sin(\alpha) & \cos(\alpha)\end{bmatrix}=\begin{bmatrix}1 & 0\\ \cot(\alpha)-\csc(\alpha) & 1\end{bmatrix}\begin{bmatrix}1 & \sin(\alpha)\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ \cot(\alpha)-\csc(\alpha) & 1\end{bmatrix} \qquad (6)$$

The phase-scrambled signal in eq. (7) can be put into the form of the FrFT through chirp multiplication, shown in eq. (8), where α and ρ are set (eq. (9)) to match the definition of the FrFT [5]. Care must be taken to evaluate the chirp function at values of ρ which correspond to the values of *k* in the signal *s*(*k*(*t*)), i.e., evaluating the chirp at *k*= csc(α)*ρ*. The image is then reconstructed using the FrFT of angle α. To scale the image by factor β, the chirp multiplier is changed to $\exp(i\pi\beta\cot(\alpha)\rho^2)$. To then reconstruct a properly "focused" image, a FrFT of new angle α_s (eq. (10)) must be performed. Setting β > 1 results in less quadratic phase across *f*(*ρ*), essentially "tricking" the FrFT into reconstructing the object within a larger FOV.

$$s(k(t))=\int \exp\left(-i2\pi\left(k(t)u+\gamma H\tau u^2\right)\right)f(u)\,du \qquad (7)$$

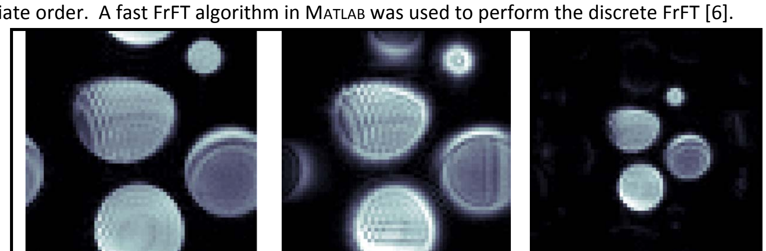
$$f_\alpha(\rho)=\sqrt{1-i\cot(\alpha)}\ \exp\left(i\pi\cot(\alpha)\rho^2\right)s(k(t)) \qquad (8)$$

$$\alpha=\cot^{-1}(-2\gamma H\tau), \qquad \rho(t)=\frac{k(t)}{\csc(\alpha)} \qquad (9)$$

$$\alpha_s=\cot^{-1}\left(\beta\cot(\alpha)\right) \qquad (10)$$

METHODS: Phase-scrambled data in the axial plane were acquired on a 3T Siemens Trio scanner using a FLASH sequence with additional TTL pulse triggers for switching a Z2 gradient insert, whose field varies as $-(X^2+Y^2)$ in the imaged plane. The Z2 gradient was pulsed for 150 μs in between slice selection and readout. The quadratic gradient strength was set high enough to permit scaling but not so high as to cause intravoxel dephasing near the periphery. Scaled images were reconstructed using a chirp multiplication and subsequent FrFT of the appropriate order. A fast FrFT algorithm in MATLAB was used to perform the discrete FrFT [6].

RESULTS: Phase-scrambled signals show the expected spectral dilation. Qualitatively this results from a convolution of the quadratic chirp with the object, the linear gradients steering the quadratic function across the FOV. Alias-free "zoomed out" magnitude images are obtained using α_s as specified in eq. (10). While previous scalable reconstructions [2] were limited to the use of second-order shim coils, leading to long *TE* times, we demonstrate using a powerful Z2 coil that scaling is possible using short pulses that can be readily incorporated into any pulse sequence.



Fourier transform of *s*(*t*) before chirp multiplication (aliased)	FrFT reconstruction of $f_\alpha(\rho)$ (after chirp) with β=1.	FrFT reconstruction with β=2.5.

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REFERENCES: [1] Ito S *et al*, MRM 2008;60:422-430. [2] Zaitsev M *et al*, ISMRM 2009, #2859. [3] Kaffanke J *et al*, JMR 2006;178:121-128. [4] Ozaktas HM *et al*, The Fractional Fourier Transform, Chichester, UK: John Wiley & Sons, 2001. [5] Parot V *et al*, ISMRM 2010 #2939. [6] Ozaktas HM, http://www.ee.bilkent.edu.tr/~haldun/fracF.m