Efficient Direct Summation Reconstruction for Radial and PROPERLLER MRI Using the Chirp Transform Algorithm

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Introduction: Direct summation reconstruction, also known as Direct Fourier transform (DFT) or Conjugate phase reconstruction in the MRI community, could reconstruct MR image from non-Cartesian data with high precision and were widely used as a reference in the evaluation for the reconstruction accuracy of different methods. However, the high computation complexity makes DFT impractical for clinical application. Up to now, the published "FFT" algorithms [1-3] for non-equispaced data do not strictly compute the DFT of nonequispaced data, but rather some approximation. In this work, an efficient algorithm for DFT using the Chirp Transform Algorithm (CTA) [4] was proposed to reconstruct MR image from non-Cartesian trajectories consisting of lines with equispaced points such as radial or PROPERLLER sampling. We demonstrate that the proposed method reconstructs image much faster than DFT while keeping the exactly same accuracy.

Theory: With the sampling density correction neglected, the DFT reconstruction can be formulated as $I(x,y)=M^{-1}\sum_{m=1}^{M} d_m(k_{x_m},k_{y_m})\exp(j2\pi(kx_{x_m}+yk_{y_m}))$ (1), where $I(x,y)$ is the reconstructed image and (x,y) denotes locations in the image, $d(k_x,k_y)$ is the sampled k-space data and (k_x,k_y) denotes k-space coordinate, $m=0,1,\cdots,M-1$ is the index and M is the number of samples. If the sampling trajectory consists of L straight lines and M_l points are sampled equispacedly on each line, Eq. (1) can be formulated as $I(x,y)=M^{-1}\sum_{l=0}^{L-1}\sum_{i=0}^{M_l-1} d_l(k_{xl_i},k_{yl_i})\exp(j2\pi(xk_{xl_i}+yk_{yl_i}))$ (2). Letting $I_l(x,y)=\sum_{i=0}^{M_l-1} d_l(k_{xl_i},k_{yl_i})\exp(j2\pi(xk_{xl_i}+yk_{yl_i}))$ (3), we can get $I(x,y)=K^{-1}\sum_{l=0}^{L-1} I_l(x,y)$ (4). This means that we can first get a temp image through performing the Fourier transform of each k-space line, and then reconstruct the final image by combing all the temp images, which can be justified by the linearity of DFT.

Given an arbitrary line l in k-space, the ith equispaced point on it can be expressed as $k_{lx_i}=k_{lx_0}+i\Delta k_{lx}, k_{ly_i}=k_{ly_0}+i\Delta k_{ly}$ (5). The calculate Eq. (3), the data on each line is zero-padded into a $M_l\times M_l$ matrix (denoted as D_l) whose diagonal elements are the sampled data. Then we can get, $I_l(x,y)=\sum_{j=0}^{M_l-1}\sum_{i=0}^{M_l-1} D_l(i,j)\exp(j2\pi(k_{lx_0}+i\Delta k_{lx})x)\exp(j2\pi(k_{y0}+j\Delta k_y)y)=\sum_{j=0}^{M_l-1}\exp(j2\pi(k_{y0}+j\Delta k_y)y)\sum_{i=0}^{M_l-1} D_l(i,j)\exp(j2\pi(k_{lx_0}+i\Delta k_{lx})x)$ (6), which means the calculation of Eq. (6) can be fulfilled by 1D DFT along the x direction and the y direction sequentially. The calculation of Eq. (6) can be accelerated by the chirp transform algorithm (CTA), which is able to compute any set of equally spaced samples of the Fourier transform on the unit circle. The CTA is based on expressing the DFT as a convolution that is computed by the FFT algorithm.

Given the 1D IDFT $I(n)=\sum_{k=0}^{K-1}F(k)\exp(j2\pi n(k_0+k\Delta k))=\exp(j2\pi nk_0)\sum_{k=0}^{K-1}F(k)\exp(j2\pi nk\Delta k), k=0,1,\cdots,K-1$ (7), since $nk=[n^2+k^2-(n-k)^2]/2$, so we get $I(n)=\exp(j2\pi(k_0+\Delta kn^2/2))\sum_{k=0}^{K-1}F(k)(e^{j2\pi\Delta k})^{k^2/2}(e^{j2\pi\Delta k})^{-(n-k)^2/2}$ (8). Letting $A(n)=e^{j2\pi(k_0+\Delta kn^2/2)}$, $g(k)=F(k)(e^{j2\pi\Delta k})^{k^2/2}$ and $h(k)=(e^{j2\pi\Delta k})^{k^2/2}$, we can get $I(n)=A(n)\{g(k)*h(k)\}_{0,\ldots,N-1}$ (9). The convolution in Eq. (9) can be implemented efficiently using two FFTs and one IFFT as illustrated in Fig. 1: $G(r)=FFT(\{g(k)\}_{0,1,\ldots,L-1})$, $H(r)=FFT(\{h(k)\}_{0,1,\ldots,L-1})$ and $\{g(k)*h(k)\}=IFFT\{G(r)H(r)\}$, where $L=N+K-1$.

The number of complex multiplication required for each step in CTA is shown in Fig.1. The time complexity of CTA for all the x-direction is $2^{-1}L\log_2 L+K(K+2^{-1}L\log_2 L+2L+N)$ and that for y-direction is $2^{-1}L\log_2 L+N(K+L\log_2 L+L+N)$. If assuming $K\approx N, L\approx 2N$, the computation complexity of 1D CTA-DFT algorithm is: $3N^2\log_2 N+13N^2+2N\log_2 N+2N\approx N^2(3\log_2 N+13)$, while that of the DFT is N^3. The acceleration factor is about 6.91 for N=256 and 12.79 for N=512.

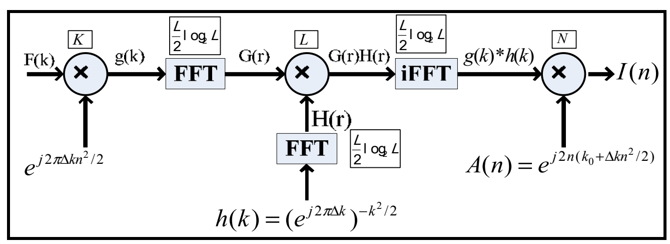
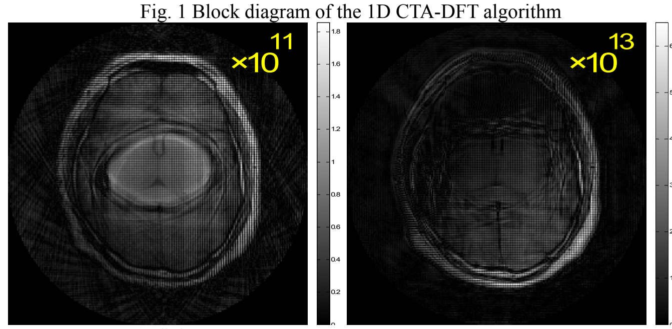


Fig. 1 Block diagram of the 1D CTA-DFT algorithm

Methods: Radial and PROPERLLER MRI data were synthesized by performing DFT on a collected MR image. The image size was 256×256. Each line contained 256 points. For Radial sampling, 432 lines were synthesized. For PROPERLLER, each strip contained 24 phase-encoding lines and 18 strips were synthesized. Sampling density correction with designed kernel from [5] was implemented. The Cooley–Tukey algorithm and FFTW (the Faster Fourier Transform in the West) were implemented for FFT in CTA respectively.

Results and Discussion: The difference image between the reconstruction using the CTA-DFT and DFT are shown in Fig. 2. For the radial trajectory, the maximum difference between the two algorithms is 1.86×10^{-10} and the mean difference is 2.36×10^{-11}. For PROPELLER trajectory, the maximum difference is 6.41×10^{-13} and the mean difference is7.35×10^{-14}.

The time costs of DFT and CTA-DFT using Cooley–Tukey algorithm and FFTW are presented in Table 1. The algorithm was implemented on a PC with Intel Core 2 Quad Q8200 2.33-GHz CPU and 8-GB DDR2 memory.Compared with DFT, time cost of the CTA-DFT using the Cooley–Tukey algorithm matchs very well with the above analysis of computation complexity. FFTW can further accelerate the CTA-DFT greatly as shown in the fourth column in Table 1.



Radial PROPERLLER

Fig. 2 Difference images between DFT and CTA-DFT

Conclusion: Based on Linearity of DFT and the CTA, this paper proposed a new algorithm to reconstruct MR image from non-Cartesian sampling consisting of straight lines such as radial and PROPERLLER. The proposed CTA-DFT algorithm explores the inherent regularity in the sampling pattern (equispaced points on a line) and is demonstrated to be significantly faster than DFT while keeping the same accuracy. Further accelerating can be attained by parallel computing using GPGPU (general purpose graphic processing unit), which owns hundreds of ALUs (Arithmetic Logical Unit).

Table 1 Time cost of reconstruction (seconds)

Dataset	DFT	CTA-DFT	
		Cooley–Tukey	FFTW
Radial	637.50	79.99	16.33
PROPELLER	649.12	79.84	18.54

Reference

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