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Bessel Fourier Orientation Reconstruction: Using Heat Equation and Multiple Shell Acquisitions to Reconstruct Diffusion Propagator

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Introduction: The estimation of the ensemble average propagator (EAP) directly from the ***q***-space DWI signals is an open problem in diffusion MRI. Diffusion spectrum imaging (DSI) [1] is one common technique to compute the EAP directly from diffusion signal, but is burdened by the large sampling required. Recently, several analytical EAP reconstruction techniques for multiple ***q***-shell acquisitions have been proposed, including Diffusion Propagator Imaging (DPI) [2] and Spherical Polar Fourier Imaging (SPFI) [3]. SPFI has only been applied, however, to low *b*-values (up to *b*=3000 s/mm²), and also requires many samples. DPI is based on the Laplace's equation estimation of diffusion signal for each shell acquisition. Viewed intuitively in terms of the heat equation, the DPI solution is obtained when the heat distribution at each shell is at a steady state.

We propose a generalized extension of DPI, calling it Bessel Fourier Orientation Reconstruction (BFOR), whose solution is based on the heat equation estimation of diffusion signal for each shell acquisition – that is, the heat distribution at each shell is no longer at a steady state. In addition to being linear and analytical, the BFOR solution also includes an intrinsic exponential smoothening term. We test our approach on synthetic data using the hybrid diffusion imaging (HYDI) [4] sampling scheme.

Theory: Consider the eigenvalue problem $\Im_x \psi_j(x) = -\lambda_j \psi_j(x)$ (1), which we use to solve the PDE $\frac{\partial}{\partial t} g(x,t) - \Im_x g(x,t) = 0, \quad g(x,0) = f(x)$ (2), where *f*(*x*) is simply the measured signal and $\Im$ is some linear operator. The problem at hand is simply a Cauchy problem, and the solution can be expressed as a linear combination of orthonormal bases: $g(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle f, \psi_j \rangle \psi_j(x)$ (3), where $e^{-\lambda_j t}$ is a smoothening term controlled by parameter *t* [5]. We take the operator of (1) to be 3D Laplacian in spherical coordinates, which allows us to solve (1) via the separation of variables. Assuming a Laplacian operator, (2) simply becomes the heat equation. Assuming the ***q***-space signal satisfies (2), then it can be expressed in terms of an orthonormal basis: $E(q,\hat{\boldsymbol{u}},t) = \sum_{n=1}^{N} \sum_{l=0}^{L} \sum_{m=-l}^{l} C_{nlm} e^{-\alpha_{nl}^2 t/\Delta^2} j_l\left(\frac{\alpha_{nl} q}{\Delta}\right) Y_l^m(\hat{\boldsymbol{u}})$ (4), where α_{nl} is n^{th} zero of l^{th} order spherical Bessel function of 1st kind j_l, Y_l^m are spherical harmonics, and $\Delta > q_{\max}$ is an upper bound at which signal becomes zero. Since we only use the even ordered spherical harmonics, the number of unknown coefficients C_{nlm} is N(L+1)(L+2)/2.

When the narrow pulse condition is met, the EAP is simply the Fourier transform of ***q***-space signal [6]. Expressing the Fourier kernel as a plane wave expansion in spherical coordinates and exploiting algebraic properties of the spherical Bessel function, an analytical expression of EAP can be derived:

$$P(\rho,\hat{r},t) = \sqrt{\frac{\pi^3 \Delta}{2\rho}} \sum_{n=1}^{N} \sum_{l=0}^{L} \sum_{m=-l}^{l} (-1)^{l/2} C_{nlm} e^{-\frac{\alpha_{nl}^2 t}{\Delta^2}} Y_l^m(\hat{r}) \frac{J_{l+1/2}(2\pi\rho\Delta)}{\left(4\pi^2\rho^2 - \frac{\alpha_{nl}^2}{\Delta^2}\right)} D_{nl} \quad (5)$$

where $D_{nl} = \sqrt{\alpha_{nl}}\left(J_{l-1/2}(\alpha_{nl}) - J_{l+3/2}(\alpha_{nl})\right)$.

Synthetic Data: We apply BFOR to simulations of crossing fiber configurations using a Gaussian mixture model: $E(\boldsymbol{q}) = \sum_{k=1}^{n} f_k e^{-\boldsymbol{q}' \boldsymbol{D}_k \boldsymbol{q}}$, where *n* denotes number of fibers and $\sum_{k=1}^{n} f_k = 1$. The HYDI sampling scheme is given in Table 1 ($q_{\max}$=1 mm⁻¹). We use a regularized Moore-Penrose pseudo-inverse scheme, where the regularization matrix is the Laplace-Beltrami diagonal matrix [7]. We first look at noise-free case and then add Rician noise with $SNR=1/\sigma$, which is defined as the ratio of maximum signal intensity of $S(0)=1$ to the standard deviation σ of complex Gaussian noise. For noise case, 1000 trials were conducted.

For simulations, we look at two fibers (equally weighted) crossing at 75°. The eigenvalues of each diffusion tensor are [1.6, 0.4, 0.4]e-3 and we use N=4, L=4, Δ=1.2 mm⁻¹, and *SNR*=20. Our results are shown in Fig. (1) and (2) for ρ=0.6, 0.8, and 1 mm. For the noise-free case, BFOR gives a nearly perfect EAP reconstruction. When *SNR*=20, BFOR underestimates the true EAP and the underlying geometry is not exact, but the fiber orientation is correct.

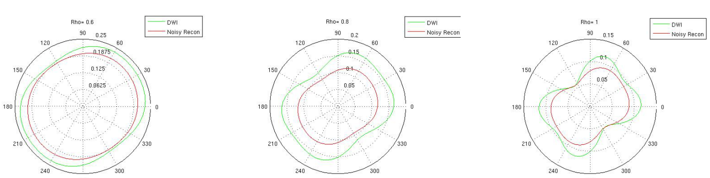

Figure 1: Noisy EAP reconstruction, *SNR*=20.

Table 1	
Ne	*b* (s/mm²)
1	0
6	375
21	1500
24	3375
24	6000
50	9375

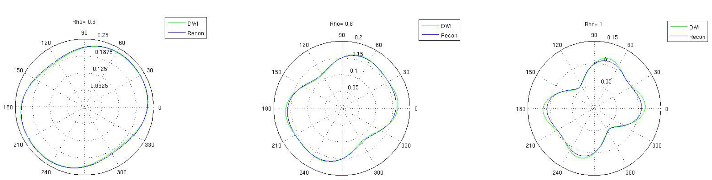

Figure 2: Noise-free EAP reconstruction.

References: [1] Weeden et al. MRM (2005). [2] Descoteaux et al. IPMI (2009). [3] Cheng et al. MICCAI (2010). [4] Wu & Alexander. Neuroimage (2007). [5] Chung et al. IEEE Trans. Med. Imaging(2007). [6] Callaghan et al. Nature (1991). [7] Descoteaux et al. MRM (2007).