# Series expansion of multi-compartment models for DCE-MRI

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**PURPOSE:** Improvements in DCE-MRI data quality are increasingly generating a need for advanced compartmental models to better describe the physiology underlying these data [1]. However, this development is severely constrained by the fact that analytical solutions exist for only the simplest multi-compartment models. Even when such solutions do exist [2], they are algebraically complicated and do not provide much physical insight. In this study an intuitive approach is presented for generating approximations to multi-compartment models for DCE-MRI with any desired accuracy. The method is illustrated by application to the two-compartment exchange model (2CXM) [2].

**METHODS**: We use the following definitions and relations. A multi-compartment model can be depicted as a *flow graph* [3], where each *node* represents a compartment, and the *arrows* represent tracer current between nodes. The *paths* of the model are the possible trajectories that a tracer particle can take in passing through the system. The *order* of a path is the number of nodes that a particle enters on the path, and the *exponent* of an arrow on a path is the number of times a particle crosses the arrow. Each node X is characterised by a *mean transit time* $T_X$, and each arrow A by a *flow* $F_A$. The *capacity* of an arrow A ending in a compartment X is the ratio of $F_A$ to the sum of all flows leaving X. The *propagator* H(t) of the system is the distribution of transit times. H(t) relates the tracer concentration $C_a(t)$ at the inlet and the flow F through the inlet to the total concentration C(t) inside the system: $C=F*C_a - F*H*C_a$ [4]. The propagator of a compartment X is $H_X(t)=\exp(-t/T_X)/T_X$, and the propagator of a series of *n* identical compartments X is $H_X^{(n)}(t)=H_X(t)\,(t/T_X)^{n-1}/(n-1)!$.

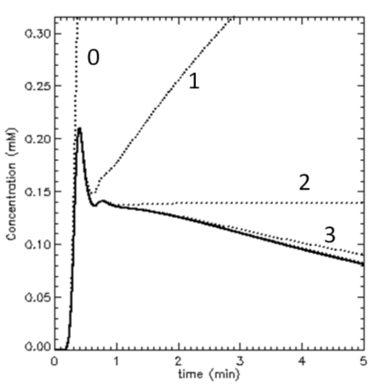

**Fig 2**. Numerical illustration for a 2CXM with E=20%, F=50ml/min/100ml, $v_p$=5% and $v_e$=10%. Shown are the exact C(t) (full line) and the approximations up to 3[d] lowest order (dotted).

**RESULTS:** We derived a graphical method for generating a convergent series expansion $H=\Sigma_i P_i H_i$, based on the principle that H is the sum of propagators $H_i$ for all possible paths i, each weighted by the probability $P_i$ that a particle follows the path i. The series is obtained in four steps, illustrated in fig 1 for the 2CXM: (a) define the system by drawing a flow graph, and labelling all nodes and the flows of each arrow; (b) calculate the capacities of each arrow; (c) graphically list the paths through the system in increasing order, and label each arrow by its exponent; (d) for each path i, calculate $P_i$ as the product of the capacities of all arrows on the path i, and $H_i$ as the convolution of the propagators of all nodes on the path i (using the formulae for $H_X$ and $H_X^{(n)}$ ). A simulation up to 3[d] order using a population-averaged $C_a$ [5] is shown in fig 2. The result shows that the series converges at relatively low orders, and that higher orders are only required at longer acquisition times.

**CONCLUSION:** Using the principles set out here, the analytical solution for arbitrary multi-compartment models can be written out to any desired accuracy by graphically listing in order the paths through the system. Since higher-order paths are increasingly unlikely, and have increasingly long transit times, the series becomes accurate at some finite order k. The method increases the physical insight in the behaviour of more complex compartmental models, and will allow more freedom and flexibility in the design of advanced models for particular applications.

**REFERENCES**: **[1]** Koh 2010 MRM Sep 21 (Epub) **[2]** Sourbron 2009 MRM 62: 205-16 **[3]** Rescigno 1963 Ann NY Acad Sci 108: 204-16 **[4]** Zierler 1965 Circ Res 16;309-321 **[5]** Parker 2006 MRM 56: 993-1000

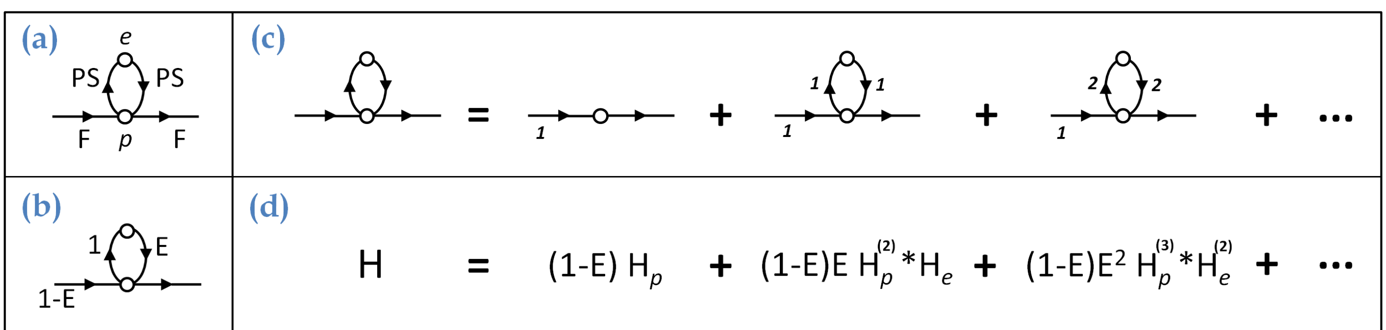


**Fig 1**. The four steps (a-d) to derive the series expansion of H, illustrated for the 2CXM. **(a)** The diagram shows the compartments plasma (p) and interstitium (e), and flows F and PS. **(b)** All capacities can be expressed in terms of the extraction fraction E=PS/(PS+F). **(c)** The three lowest-order paths: the 1[st] order path passes through *p* and then leaves the tissue; the 2[nd] order passes consecutively through *p*, *e*, *p* before leaving; the 3[d] order passes through *p, e, p, e, p*, and so on. **(d)** Probabilities and propagators for each of the paths in (c). Note that the graphical exponents in (c) directly translate to the algebraic exponents in (d).