Improved k-t FOCUSS using a sparse Bayesian learning

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Introduction: In dynamic MRI, spatio-temporal resolution is a very important issue. Recently, compressed sensing approach has become a highly attracted imaging technique since it enables accelerated acquisition without aliasing artifacts. Our group has proposed an l_1-norm based compressed sensing dynamic MRI called k-t FOCUSS which outperforms the existing methods. However, it is known that the restrictive conditions for l_1 exact reconstruction usually cost more measurements than l_0 minimization. In this paper, we adopt a sparse Bayesian learning approach to improve k-t FOCUSS [1, 2] and achieve l_0 solution. We demonstrated the improved image quality using cardiac cine imaging.

Theory: From compressed sensing perspective, the sparse approximation for dynamic MR imaging problem can be stated as follows:

$$\min \|\rho\|_1, \quad \text{subject to} \quad \|v - F\rho\|_2 \le \varepsilon, \tag{1}$$

where ρ is a vector stacking sparse unknown sparse signals with size of M and v is a vector stacking under-sampled k-t measurements with size of N $(N < M)$. Then, the sensing matrix F is defined by down-sampled Fourier transform along phase encoding direction multiplied with sparsifying transform. Solving Eq. (1) can be recast in Bayesian terms by applying $p(v \mid \rho) \propto \exp[-1/\lambda \|v - F\rho\|_2^2]$ and a priori distribution $p(\rho) \propto \exp[-\|\rho\|_p^p]$. A priori distribution for Eq. (1) corresponds to $p = 1$. Then, Eq. (1) can be interpreted as a *maximum a posteriori* (MAP) estimation as follows:

$$\hat{\rho} = \arg\max p(v \mid \rho) p(\rho) = \arg\max p(\rho \mid v). \tag{2}$$

Then, using the expectation maximization (EM) algorithm with a set of latent variables θ_n related to ρ, Eq.(2) can be solved as follows:

$$E-step: \theta_n(i) = |\hat{\rho}_n(i)|^{2-p}, \quad M-step: \hat{\rho}_{n+1} = \Theta_n F^H (F\Theta_n F^H + \lambda I)^{-1} v, \text{ where } \Theta_n = diag(\theta_n). \tag{3}$$

As $p(\rho)$ is a priori distribution for the unknown ρ, as $p \to 0$, $p(\rho)$ has a higher probability at $\|\rho\|_p \to 0$. This means that sparse solution has a higher likelihood as desired. However, if $p < 1$, the optimization is not convex so that the number of local minima combinatorially increases. Therefore, we cannot guarantee that the l_p $(0 < p < 1)$ minimization always solves a sparser solution than l_1 minimization.

To address this problem, sparse Bayesian learning (SBL) [3] uses an empirical prior, which is a flexible priori distribution dependent on a set of unknown hyperparameters that must be estimated from the data. More specifically, instead of specifying l_p norm, SBL imposes sparsity for each unknown pixel by assuming zero mean Gaussian distribution with an unknown variance θ. Then, the posterior density of ρ is obtained with

$$p(\rho \mid v;\theta) = p(\rho, v;\theta) \Big/ \int p(\rho, v;\theta) d\theta = N(\hat{\rho}, \Sigma), \tag{4}$$

with mean and covariance given by:

$$\hat{\rho} = \Theta F^H (\lambda I + F\Theta F^H)^{-1} v, \tag{5}$$

$$\Sigma = \Theta - \Theta F^H (\lambda I + F\Theta F^H)^{-1} F\Theta, \qquad \text{where } \Theta = diag(\theta). \tag{6}$$

Then, when $\theta(i) = 0$, $\hat{\rho}(i)$ is zero with probability of 1 as desired from Eq. (5). Therefore, sparsity of ρ is determined by the sparsity of hyperparmeters θ. In other words, estimating the hyperparameters θ is equal to a model selection of a priori distribution. In this context, estimating sparse hyperparameters is the only problem we have to address so that the unknown signal ρ can be integrated out. Then, the maximum likelihood of θ can be calculated by minimizing the following cost function:

$$L(\theta) = -2\log \int p(v \mid \rho) p(\rho;\theta) d\rho = -2\log p(v;\theta) = \log|\lambda I + F\Theta F^H| + v^H (\lambda I + F\Theta F^H)^{-1} v. \tag{7}$$

Taking a derivative of Eq. (7) with respect to θ, the minimizer can be achieved with the following fixed point iteration:

$$\theta_n(i) = |\hat{\rho}_n(i)|^2 \Big/ (1 - \theta_{n-1}(i)^{-1} \Sigma_n(i,i)), \tag{8}$$

where $\hat{\rho}_n$ and $\Sigma_n(i,i)$ are updated by mean and diagonal components of variance of Eq. (4). Finally, the optimal solution $\hat{\rho}$ is achieved by Eq. (5) when θ_n converges. Even if SBL does not appear to directly minimize l_0 norm of unknown signals ρ, it is proved that the globally optimal solution of SBL is equal to the solution of l_0 minimization of ρ [3].

Interestingly, the only difference between k-t FOCUSS and SBL comes from the update equation of θ_n. Furthermore, if the denominator of Eq. (8) is a constant, the update rule of SBL becomes exactly same with that of k-t FOCUSS with $p = 0$. However, the non-uniform denominator in Eq. (8) enables the size of local minima set of SBL to be much smaller than that of k-t FOCUSS for $p < 1$. Hence, SBL can help improve the image quality.

Results: We have acquired 25 frames of full k-space data from a cardiac cine of a patient at a 1.5 T Philips scanner. To evaluate the performance of SBL, 8-fold and 13-fold down sampled k-t measurements were used for reconstruction. In both cases, we observed that SBL reconstructs finer image structures while the results of k-t FOCUSS show blurred image features. The MSE plots also show smaller errors in k-t SBL at all of the time points.

Conclusions: This paper described a novel MR imaging algorithm derived from an empirical Bayesian method. Instead of specifying l_p $(p \le 1)$ norm, assuming a flexible prior which can be updated from measured data, more accurate sparse solution was obtained.

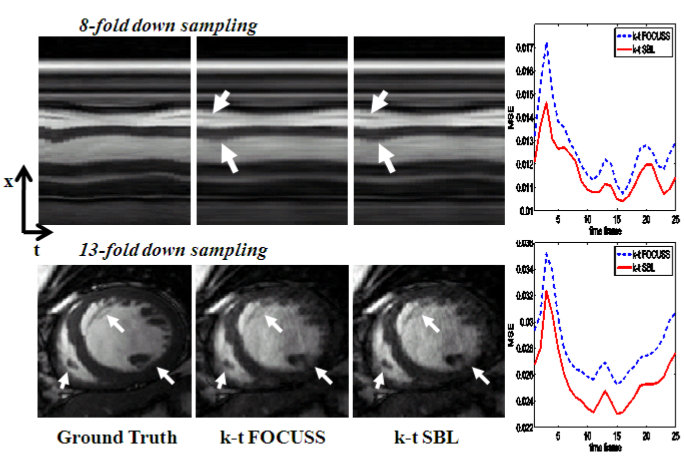

Figure 1. In vivo cardiac cine imaging using k-t FOCUSS and k-t SBL and MSE plots

References: [1] Jung et al. PMB vol.52, June 2007. [2] Jung et al. MRM vol.61, Jan 2009. [3] Wipf et al. IEEE Trans. Sig. Proc. vol. 52, Aug 2004.