Wavelet-based Compressed Sensing using Gaussian Scale Mixtures

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Introduction: A novel theory, called Compressed Sensing (CS) [1, 2], has demonstrated that MR images can be successfully reconstructed from a small number of *k*-space measurements [3]. The practical impact and success of CS in imaging applications can be attributed to the fact that most signals of practical interest have sparse representations in a transform domain. While initial CS techniques assumed that the sparsity transform coefficients are independently distributed, recent results indicate that dependencies between transform coefficients can be exploited for improved performance [4]. In this paper, we propose the use of a Gaussian Scale Mixture (GSM) model for exploiting the dependencies between wavelet coefficients in CS MRI. Our results indicate that the proposed model can significantly reduce the reconstruction artifacts in wavelet-based CS MRI.

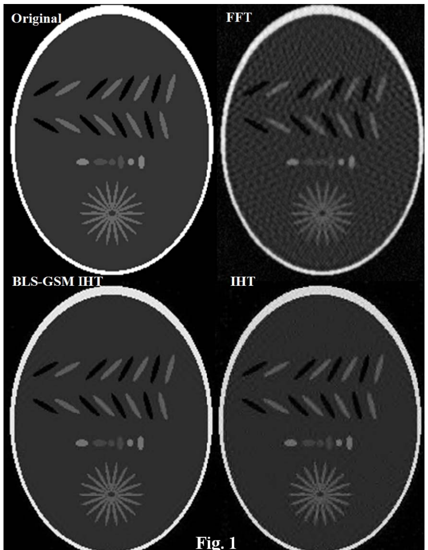

Fig. 1

Fig. 2

Theory: The wavelet-based CS MRI can be represented as the following minimization problem: $\min_{\mathbf{x}} \|\mathbf{b}-\mathbf{A}\mathbf{x}\|_2^2$ s.t. $\|\mathbf{x}\|_0 \le M$, where $\mathbf{x}$ denotes wavelet coefficients of the image to be reconstructed, $\mathbf{b}$ is the undersampled *k*-space measurements, and $\mathbf{A}=\mathbf{F}_u\boldsymbol{\psi}^{-1}$ is a matrix representing the sequential application of the inverse wavelet transform $\boldsymbol{\psi}^{-1}$ and the undersampled Fourier transform $\mathbf{F}_u$. Iterative Hard Thresholding (IHT) [5] is a simple CS technique that solves this problem through the following iterative algorithm: $\mathbf{x}^{n+1}=H_M(\mathbf{x}^n+\mu\mathbf{A}^H(\mathbf{b}-\mathbf{A}\mathbf{x}^n))$, where $\mathbf{x}^n$ refers to the solution at iteration n and H_M is a nonlinear operator that retains the M largest coefficients. μ is the optimal step size to minimize errors in each iteration [5]. In image compression and denoising, prior information about the structure of wavelet coefficients has been utilized successfully [6]. Our goal in this work is to exploit such structure within the IHT framework.

Method: In IHT, estimates of the wavelet coefficients are formed at each iteration. The key idea in this work is to include an additional step (at each iteration) to refine these estimates based on a model which exploits dependencies between wavelet coefficients. We adopt a model which has been successfully employed in wavelet based image denoising applications [7]. Let $\tilde{\mathbf{x}}$ denote the observed (noisy) coefficients in a neighborhood. We assume that the observed coefficients can be expressed as $\tilde{\mathbf{x}}=\mathbf{y}+\mathbf{w}$ where $\mathbf{y}$ is a vector containing the uncorrupted coefficients and $\mathbf{w}$ is a vector which accounts for both estimation noise and aliasing artifacts. For simplicity, we model $\mathbf{w}$ as zero-mean Gaussian with covariance matrix $\mathbf{C}_{\mathbf{w}}$. The wavelet coefficients $\mathbf{y}$ are modeled using a Gaussian Scale Mixture (GSM) such that $\mathbf{y}=\sqrt{z}\mathbf{u}$, where z is an independent scalar random variable and $\mathbf{u}$ are zero-mean Gaussian with covariance matrix $\mathbf{C}_{\mathbf{u}}$. Using this model, the Bayes least squares estimate of a coefficient in the center of a particular neighborhood y_C is given by $\mathrm{E}\{y_C|\tilde{\mathbf{x}}\}=\int_0^\infty \mathrm{E}\{y_C|\tilde{\mathbf{x}},z\}p(z\,|\,\tilde{\mathbf{x}})dz$. We refer the reader to [7] for a detailed explanation on computation of quantities such as $\mathbf{C}_{\mathbf{u}}$, $p(z\,|\,\tilde{\mathbf{x}})$, and $\mathrm{E}\{y_C|\tilde{\mathbf{x}},z\}$ in this framework. Denoting this modeling and estimation algorithm with the operator $D_{BLS-GSM}$, the improved IHT algorithm can be now stated as: $\mathbf{x}^{n+1}=H_M(D_{BLS-GSM}(\mathbf{x}^n+\mu\mathbf{A}^H(\mathbf{b}-\mathbf{A}\mathbf{x}^n)))$. Experiments were carried out on both a computer generated phantom and *in vivo* data. The *k*-space data for the phantom (256×256 pixels) were obtained by sampling *k*-space along 100 radial views with 256 points along each view. The *in vivo* dataset was obtained by selecting one sagittal slice (256×256 pixels) out of a 3D brain dataset. The *k*-space data for the *in vivo* dataset were obtained by sampling *k*-space along 130 radial views with 256 points along each view. The orthonormal Daubechies wavelet with 3 vanishing moments and 5 levels of decomposition was used as the sparsity transform in IHT. Overcomplete steerable pyramid wavelet was used during statistical modeling step. The stopping criterion for IHT was set to the *l*-2 norm of residual being less than 10^{-16}.

Results: Fig. 1 shows the original phantom image, the image obtained using FFT, the image obtained by the proposed technique (BLS-GSM IHT), and the image obtained using the conventional IHT. While both IHT techniques significantly reduce undersampling artifacts, BLS-GSM IHT is able to further reduce artifacts both along edges and in smooth areas of the image. Sections of the reconstructed images are shown in Fig. 2 to enable closer inspection of the reconstruction artifacts. Fig. 3 shows the original image, the BLS-GSM IHT reconstructed image, and the conventional IHT reconstructed image for the *in vivo* dataset. As the enlarged sections of these images (Fig. 4) illustrate, the BLS-GSM IHT technique eliminates the residual artifacts present in the IHT reconstructed image. In addition to improved image quality, the proposed technique reduces reconstruction time compared to the conventional IHT algorithm as well. While the conventional IHT algorithm required 188 iterations to converge for the *in vivo* dataset, BLS-GSM IHT required only 14 iterations.

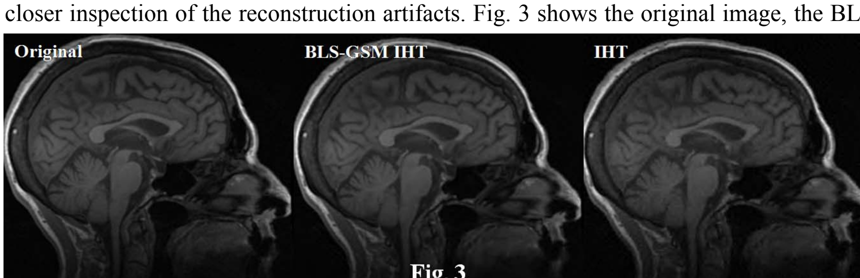

Fig. 3

Fig. 4

Conclusions: A novel CS MRI technique that exploits dependencies between wavelet coefficients is introduced. The proposed method significantly reduces residual reconstruction artifacts and reconstruction time. The proposed framework can also be extended to CS MRI techniques other than IHT as well.

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References: [1] Candes E *et al.* IEEE Transactions on Information Theory (2006). [2] Donoho D IEEE Transactions on Information Theory (2006). [3] Lustig M *et al.* MRM (2007). [4] Baraniuk RG *et al.* (2008). [5] Blumensath T *et al.* The Journal of Fourier Analysis and Applications (2008). [6] Mallat S, a Wavelet Tour of Signal Processing, academic press, 2009. [7] Portilla J *et al.* IEEE Transactions on Image Processing (2003).