Statistical noise model in GRAPPA-reconstructed images

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Noise in Parallel Imaging

Parallel MRI (pMRI) techniques extended the applicability of multiple-coil systems by increasing the acquisition rate via subsampled acquisitions of the **k**-space data. Many reconstruction methods have been proposed in order to suppress the aliasing and underlying artifacts created by the subsampling. Dominant among these are SENSE and GRAPPA. One of the effects in accelerated pMRI acquisitions is a significant change in the noise model, which depends on reconstruction scheme. From a statistical point of view, it is known that, if no subsampling of the **k**-space is done, the composite magnitude signal from several coils can be modeled as a non-central Chi distribution [Const97]. When **k**-space is subsampled, the noise power in-fact varies across the reconstructed image, and yet (maybe?) different for each receiving coil. Depending on the way the information from each coil is combined (i.e, depending on the reconstruction method), the statistics of the image may no longer follow the non-central Chi statistics. In SENSE, for instance, noise in the **x**-space is known to be Rician but non-stationary, i.e. the noise power varies from point to point [Thumb07]. Some authors suggest that GRAPPA-reconstructed images may follow a non-central Chi distribution [Thumb07], but no thorough study has been done.

The great importance of the characterization of noise statistics in MRI has been extensively reported in literature. Many filtering, noise estimation or diffusion tensor estimation methods for conventional (non-parallel) MRI rely on the Rician noise model. However, during the last years some techniques have been developed assuming a non-central Chi distribution of noise. As an example, in [TV09] authors prove that the Weighted Least Squares method used in DTI to estimate the diffusion tensor is also valid under the non-central Chi assumption.

In [Breuer09] authors present a novel g-factor derivation for GRAPPA, and the variance of noise in each coil calculated as

$$\sigma_k^2(\mathbf{x}) = |\mathbf{\Omega}|\sum_{l=1}^{L}|W_{kl}(\mathbf{x})|^2(\sigma_l^2)^{red} + 2|\mathbf{\Omega}|\sum_{l=1}^{L}\sum_{m=l+1}^{L}|W_{kl}(\mathbf{x})W_{km}(\mathbf{x})|(\sigma_{l,m}^2)^{red} \qquad (1)$$

As a consequence, the variance of noise in each coil will be Gaussian but non-stationary. If the reconstruction is made using Sum of Squares (SoS) [Const97] the composite magnitude image will follow a non-central Chi distribution only if the variance of noise in one pixel is the same for all coils, and no correlation exists between them. We have found that in practice, This is not the case; the non-central Chi model does not hold in general for GRAPPA reconstructions. However, under certain conditions (which we present below?) the variance of noise is nearly constant for all positions and coils, so a non-central Chi model may adequately fit the data, like the Rician model does for conventional MRI.

Non-central Chi model in GRAPPA: experimental results

Here we present an experiment supporting this claim. Assuming a simplified model in which the variance of noise in the k-space is originally the same in all the coils, and no correlation exists, then Eq 1can be reduced to give the power of noise in each coil as $\sigma_k^2(\mathbf{x}) = \sigma_n^2\ |\Omega|\sum_l\ |W_{kl}(\mathbf{x})|^2$. From the definition of the g-factor in each coil $g_k(\mathbf{x})$, [Breuer09]?, it is trivial to show the following relation between its Coefficient of Variation (CV) and that of $\sigma_k^2(\mathbf{x})$ (power of noise in each coil): $CV\{g_k(\mathbf{x})^2\}=CV\{\sigma_k^2(\mathbf{x})\}$. So the study of the CV of $g_k(\mathbf{x})^2$ is equivalent to the study of the CV of $\sigma_k^2(\mathbf{x})$, but the former is easier to interpret. In Fig. 1 the spatial map of the CV is depicted for a pMRI acquisition with simulated acceleration, for different acceleration rates. From the image one can extract two main conclusions:

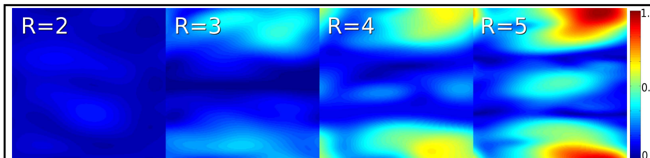

Fig. 1: CV between coils of noise power σ_n^2 for different acceleration rates from an 8 coils acquisition, GE Signa 1.5T EXCITE scanner, FSE Pulse Sequence, TR=500 ms, TE= 13.8 ms, matrix size= 256x256, FOV=20x20 cm, slick thickness= 5 mm. 15 ACS lines are considered for weights determination.

1. Although the variance of noise varies with **x**, the spatial change/gradient/fluctuation is rather slow; in the worst scenario, a certain local homogeneity can be assumed that assures a good behavior of neighborhood-based filters.
2. If there is no acceleration (R=1), then CV=0. As the acceleration rate increases, the values of the CV of noise between coils are initially low, but grow larger proportional to the acceleration rate. At high acceleration rates, the final image will differ from the non-central Chi model.

Following, we study the variation of the noise power across the receiving coils. The analysis in terms of the CV is used to characterize whether or not a non--central Chi model can be accurately fitted in each particular situation. To that end, the following (relative) Mean Squared Error (rMSE) is measured

$$\text{rMSE} = \frac{\int_{\infty}^{\infty}|G_{X_L}(w)-\widetilde{G_{X_L}}(w)|^2dw}{\int_{\infty}^{\infty}|\widetilde{G_{X_L}}(w)|^2dw}, \qquad (2)$$

By virtue of Parseval's theorem, it may be written in terms of the characteristic functions (CF) of both distributions: G_{XL} is the CF from the real distribution of the composite magnitude image obtained by SoS (without the square root, to simplify the analysis), and $\breve{G}_{XL}$ is the CF of an *equivalent* non-central Chi squared. This error is calculated and plotted in Fig. 2 as a function of the CV of σ_n^2 for different SNR.

Consider the range of CV values given by the brain data sets in the previous experiment (Fig. 1): for R=2 and R=3 CV is under 0.5. The relative errors are practically null for all SNR and coil configurations in Fig. 2. Generalizing this result, it may be concluded that the non--central Chi model may be assumed for GRAPPA reconstructed images of the brain without any loss of generality. On the other hand, for higher values of the CV, the error in the approximation rapidly increases. In particular, for values ranging from 1.0 to 1.4, the error would be appreciable even for relatively high SNR. For L=8 receiving coils, and SNR ranging in [2,4] (which may be quite realistic in diffusion weighted images), the relative error can be over 1%. For high SNR, both the true PDF and the non-central Chi squared approximation tend to a Gaussian, so the error remains small.

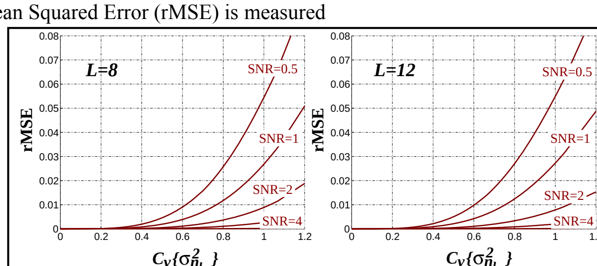

Fig. 2: Relative errors in the PDF for the non-central Chi squared approximation, as a function of the CV of the noise power at each coil. Curves are parametrized depending on the SNR. Systems with 8 (left) or 12 (right) receiving coils are considered.

Conclusions

The noise distribution in each coil after GRAPPA reconstruction is known to be Gaussian and non-stationary (i.e. spatially dependant), and correlation exists between coils. As a consequence, a non-central Chi is not an adequate noise distribution model for the final composite magnitude image. However, the non-central Chi model can be used as an approximation under certain realistic assumptions: the correlation between coils is small and the CV of the power of noise between the coils is small.

[Breuer09] Breuer *et al.* General formulation for quantitative g-factor calculation in GRAPPA reconstructions, *MRM* 62:739-746 (2009)
[Const97] Constantinides *et al.* Signal-to-noise measurements in magnitude images from NMR phased arrays, *MRM* 38:852-857 (1997)
[Thumb07] Thumberg *et al.* Noise distribution in SENSE- and GRAPPA-reconstructed images: a comp. simulation study, *MRI* 25:1089 (2007)
[TV09] Tristán-Vega, *et* al. Bias of least squares approaches for diffusion tensor estimation from array coils in DT–MRI. *Lec. Notes on Comp. Science, MICCAI* 2009.