Generalized formalism of the Extended Phase Diagram and computational applications including an MRI simulator.

G. Palma[1,2], M. Comerci[2], A. Prinster[2,3], M. Quarantelli[2], and B. Alfano[2]

[1]ESAOTE s.p.a., Naples, Italy, [2]Biostructure and Bioimaging Institute, National Research Council, Naples, Italy, [3]"S.D.N." Foundation, Naples, Italy

Introduction: MRI rests on the capability to detect the ***emf*** induced in a receiver coil by a set of spin isochromats precessing at slightly different frequencies around the main external magnetic field. The amplitude of the received signal depends not only on N(H), T_1 and T_2, but also on the configuration of the isochromat fan produced by the pulse sequence applied to the sample. The standard approach to give a geometric insight into the spin dephasing is provided by the direct integration of the Bloch equations applied to each isochromat. However that method proves to be inadequate to handle the complexity of the fans produced even by a simple train of RF pulses (see Fig. 1). A first powerful attempt to supply a quantitative description of the signal due to a sequence of α RF–pulses, with interleaved *identical* dephasing (gradient) lobes (under the assumption of gradient lobes evenly dephasing the isocromats) has been suggested (Extended Phase Diagram) [J Magn Reson 1988;78:397] [Concepts Magn Reson 1999;11:291].

Here we propose a rigorous enhancement of the formalism which accounts for the complexity of the effects produced by an arbitrary pulse sequence (no assumption on gradient lobe area and RF–pulse spacing will be made), and present the implementations of the algorithm within two complementary frameworks: Mathematica®, for analytical computation of signal equations, and MATLAB®, for numerical evaluation of signal evolution and the realization of a robust and fast MRI simulator.

Formalism: We call *phase state* a microstate (specific detailed configuration) of the isochromat sample (V). Adopting the axial representation of the magnetic moment [$\mu_+ = \mu_x + i\mu_y,\ \mu_- = \mu_x - i\mu_y,\ \mu_z$], we label each kind (transversal and longitudinal) of elementary state (F and Z) by means of two indexes (${}_i\mu$ being the magnetic moment of the i-th isochromat):

$$\bar{F}_{n,t} = \left\{ {}_i\vec{\mu} = \left(\begin{array}{c} \|{}_i\vec{\mu}\| \exp\left[i\gamma\left(n\vec{A}_G \cdot {}_i\vec{r} + {}_i\Delta B_0 t \right)\right] \\ \|{}_i\vec{\mu}\| \exp\left[-i\gamma\left(n\vec{A}_G \cdot {}_i\vec{r} + {}_i\Delta B_0 t \right)\right] \\ 0 \end{array} \right) \middle| i \in V \right\}; \quad \bar{Z}_{n,t} = \left\{ {}_i\vec{\mu} = \left(\begin{array}{c} 0 \\ 0 \\ \|{}_i\vec{\mu}\| \exp\left[i\gamma\left(n\vec{A}_G \cdot {}_i\vec{r} + {}_i\Delta B_0 t \right)\right] \end{array} \right) \middle| i \in V \right\}$$

The first, usual, index (n) stands for the number of gradient lobes applied to the sample, while the new one (t) represents the time of T_2' dephasing occurred after initial isochromat alignment. This way we are able to write a generic state produced by the application of an arbitrary pulse sequence as linear combination of basis states:

$$\bar{F} \equiv \left\{ {}_i\vec{\mu} \middle| i \in V \right\} = \sum_t \sum_{n=-\infty}^{\infty} F_{n,t}\bar{F}_{n,t} + Z_{n,t}\bar{Z}_{n,t},$$

with the usual reality condition:

$$Z_{-n,-t} = Z^*_{n,t}$$

When an α RF-pulse (with phase φ) is applied to the ensemble, the populations of the states changes according to the representation of a specific element of $SO(3)$ on the vector space generated by phase states; similarly, the dephasing and relaxation occurring during a gradient pulse (of area mA_G and duration Δt) between an RF-pulse and the following one may be easily understood in terms of a winding up of the states. In formulae:

$$\begin{pmatrix} F_{n,t} \\ F^*_{-n,-t} \\ Z_{n,t} \end{pmatrix} \xrightarrow{RF(\alpha,\phi)} \begin{pmatrix} \cos^2\frac{\alpha}{2} & e^{2i\phi}\sin^2\frac{\alpha}{2} & -ie^{i\phi}\sin\alpha \\ e^{-2i\phi}\sin^2\frac{\alpha}{2} & \cos^2\frac{\alpha}{2} & ie^{-i\phi}\sin\alpha \\ -\frac{i}{2}e^{-i\phi}\sin\alpha & \frac{i}{2}e^{i\phi}\sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} F_{n,t} \\ F^*_{-n,-t} \\ Z_{n,t} \end{pmatrix}; \quad F_{n,t} \xrightarrow{Grad(m,\Delta t)} F_{n-m,t-\Delta t}e^{-\Delta t/T_2}; \quad Z_{n,t} \xrightarrow{Grad(m,\Delta t)} \begin{cases} Z_{n,t}e^{-\Delta t/T_1}, & \forall (n,t) \neq (0,0) \\ Z_{0,0}e^{-\Delta t/T_1} + M_0(1-e^{-\Delta t/T_1}), \end{cases}$$

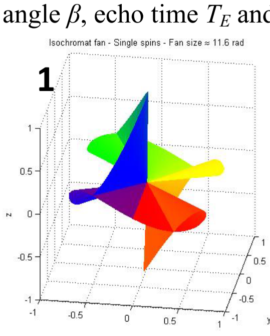
It is noteworthy that this formalism applies plainly even in the case of partially dephased isochromats, thus allowing, for instance, to properly point out non-trivial dependences of the signal on T_2' (*e.g.* all $F_{0,t}$ states, whenever m/Δt varies throughout the sequence). Therefore, the echo signal ***S*** can be written as:

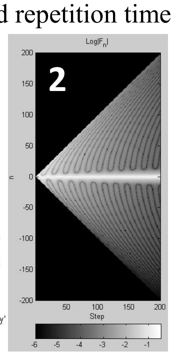
$$S = \sum_t F_{0,t} e^{-t/T_2'}$$

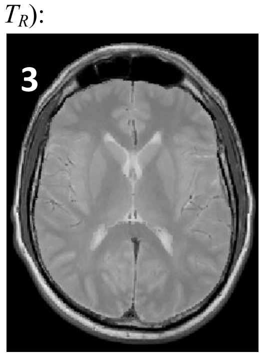
Computational tools: The Mathematica® and MATLAB® programs that implement the above algorithm, evolve the state populations according to the pulse sequence provided by the user in the shape of a structure.

In the first case, the realized tool supplies in a fully automated way the analytic signal equation. As a demonstrative example, we report the output relative to the well known spin-echo sequence (flip angle α, inversion angle β, echo time T_E and repetition time T_R):

$$S \propto \rho e^{-\frac{T_E}{T_2}} \frac{\exp\left(T_R/T_1\right) - \exp\left(T_E/2T_1\right) + \left[\exp\left(T_E/2T_1\right) - 1\right]\cos\beta}{\exp\left(T_R/T_1\right) - \cos\alpha\cos\beta} \sin\alpha \sin^2\frac{\beta}{2}$$







On the other hand, an example of the versatility of the developed MATLAB® code is given in Figs. 2-3.

Conclusions: we present a formalism allowing to simulate MR signal behavior under reduced assumptions, thus eliminating potential errors which may arise from T2*-related dephasing occurring during inter-RF pulse intervals.

Fig. 1*: Isochromat spreading as a result of 5 90° pulses.* ***Fig2****: Evolution of transversal state populations in a SSFP sequence (α=20°, T_R=15 ms, T_1=1 s, T_2=0.5 s).* ***Fig. 3****: Corresponding simulated PSIF image based on relaxometric data obtained from N(H), T1 and T2 maps of a digital phantom (ISMRM 2007, EPOS 3698).*