Adiabatic pulses revisited through averaging

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Introduction: The use of adiabatic passages is popular in magnetic resonance as a result of their robustness to the RF field inhomogeneities [1, 2], thus a computationally efficient method of solving the Bloch equation for adiabatic pulses is important for design purposes. Here, the Bloch equation is scaled and subsequently averaged to find the magnetization behaviour in a straight forward way with negligible error for adiabatic passages. The novel framework presented here may be used to optimise the adiabatic modulation functions [1, 3].

Theoretical Results: Radio frequency waveforms in MR systems are generated by piece-wise constant signals (Fig. 1), and pulse design is based on the Bloch equation. By neglecting the relaxation time constants during the excitation period, the Bloch equation in the classical rotating frame of reference is $\dot{\mathbf{M}}' = \mathbf{\Omega}'\mathbf{M}'$ where $\mathbf{M}'=[\mathrm{M}_{x'}\ \mathrm{M}_{y'}\ \mathrm{M}_{z'}]^T$, $\mathbf{\Omega}'=\begin{pmatrix} 0 & \Delta\omega & 0 \\ -\Delta\omega & 0 & u(t) \\ 0 & -u(t) & 0 \end{pmatrix}$, $u(t) = \omega_1(t) = \gamma B_1^e(t)$, and $\Delta\omega$ represents an isochromat that can be time dependent. To scale the Bloch equation, we define $\mathbf{M}'' = \mathbf{\sigma}_2(t)\mathbf{\sigma}_1(t)\mathbf{M}'$ in which $\mathbf{\sigma}_1(t) = \begin{pmatrix} \cos\alpha(t) & 0 & -\sin\alpha(t) \\ 0 & 1 & 0 \\ \sin\alpha(t) & 0 & \cos\alpha(t) \end{pmatrix}$,

$\mathbf{\sigma}_2(t) = \begin{pmatrix} \cos\int_0^t a(\tau)d\tau & -\sin\int_0^t a(\tau)d\tau & 0 \\ \sin\int_0^t a(\tau)d\tau & \cos\int_0^t a(\tau)d\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\alpha(t) = \arctan(u(t)/\Delta\omega)$, and $a(t) = \sqrt{u(t)^2 + \Delta\omega^2}$. It is possible to show that the Bloch equation can be written as $\dot{\mathbf{M}}'' = \mathbf{\Omega}''\mathbf{M}''$ where $\mathbf{\Omega}''=\begin{pmatrix} 0 & 0 & -f(t) \\ 0 & 0 & -g(t) \\ f(t) & g(t) & 0 \end{pmatrix}$ in which $f(t) = \dot{\alpha}(t)\cos\int_0^t a(\tau)d\tau$, and $g(t) = \dot{\alpha}(t)\sin\int_0^t a(\tau)d\tau$.

For a constant input the above equation simplifies to $\dot{\mathbf{M}}'' = 0$. For a piecewise constant RF excitation (Fig. 1), we may write

$\mathbf{\Omega}''=\begin{pmatrix} 0 & 0 & -f_d(t) \\ 0 & 0 & -g_d(t) \\ f_d(t) & g_d(t) & 0 \end{pmatrix}$ where $f_d(t) = \sum_{i=0}^{N-1}(\alpha_{i+1}-\alpha_i)\delta(t-i\Delta T)\cos\sum_{j=0}^{i} a_j\Delta T$, and $g_d(t) = \sum_{i=0}^{N-1}(\alpha_{i+1}-\alpha_i)\delta(t-i\Delta T)\sin\sum_{j=0}^{i} a_j\Delta T$,

with $\alpha_0 = 0$, $a_0 = 0$, and δ the Dirac delta function. Our primary result is that the solution to the averaged scaled Bloch equation, $\dot{\mathbf{M}}''_{\mathrm{avg}} = \mathbf{\Omega}''_{\mathrm{avg}}\mathbf{M}''_{\mathrm{avg}}$ which is a linear time invariant system, may be written as

$\dot{\mathbf{M}}''_{\mathrm{avg}} = \exp(\mathbf{A})\mathbf{M}_0$ where $\mathbf{M}_0$ is the initial condition of magnetization,

$\mathbf{A}=\begin{pmatrix} 0 & 0 & -f_{davg} \\ 0 & 0 & -g_{davg} \\ f_{davg} & g_{davg} & 0 \end{pmatrix}$, $f_{davg} = \sum_{i=0}^{N-1}(\alpha_{i+1}-\alpha_i)\cos\sum_{j=0}^{i} a_j\Delta T$, and

$g_{davg} = \sum_{i=0}^{N-1}(\alpha_{i+1}-\alpha_i)\sin\sum_{j=0}^{i} a_j\Delta T$. Thus, the solution to the averaged scaled Bloch equation may be achieved by a single matrix exponential. Since for adiabatic passages $|\dot{\alpha}(t)| << a(t)|$ always holds the error of the averaged solution can be shown analytically to be negligible.

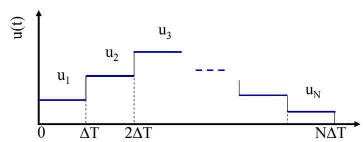

Fig. 1: A typical piece-wise constant excitation pattern.

Simulation Results: Fig. 2 shows the simulation results for a typical Adiabatic Full Passage (AFP) chirp signal with amplitude of 117 μT and frequency sweep of 2×10^4 (1-800t) Hz as in [1]. As shown in this figure magnetisation behaviour from the scaled Bloch equation is a slowly varying signal compared to the magnetisation in the classical rotating frame of reference. The error at the end of the pulse excitation period is less than three percent (Fig 2.d).

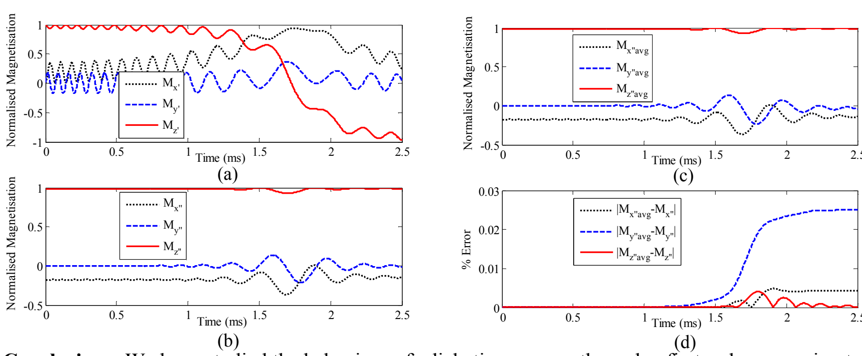

Fig. 2: Simulation results of a 4 kHz isochromat for a chirp adiabatic passage with amplitude of 117 μT and frequency sweep of 2×10^4 (1-800t) Hz in (a) the classical rotating frame of reference, (b) the scaled Bloch equation, (c) the averaged scaled Bloch equation, and (d) the error between the averaged solution and the exact solution.

Conclusions: We have studied the behaviour of adiabatic passages through a first order averaging technique used the in nonlinear dynamical systems theory [4]. A surprising result is that in this novel representation of the Bloch equation, the solution is given by a single matrix exponential, and is therefore an extremely computationally efficient method. Simulation results demonstrate the negligible error that can be proven analytically. The method can be directly applied to aid the design of adiabatic passages in MRI.

References: [1] M. Garwood, and L. Delabarre, JMR, 2001, 153: 155-177. [2] S. Taheri, and R. Sood, JMRI, 2006, 24: 51-59. [3] S. Michaeli *et al.*, JMR, 2006, 181: 135-147. [4] J. Sanders, *et al.*, *Averaging Methods in Nonlinear Dynamical Systems,* Springer-Verlag, 2007.