

# Spectral Decomposition of a 4-rank Tensor and Applications to Generalised Diffusion Tensor Imaging

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## Introduction:

In 2003 Ozarslan *et al* introduced Generalised DTI, and showed how the data from a diffusion MRI experiment can be described by a higher rank tensor [1]. In 2005 the same authors proposed two generalised anisotropy metrics that can be calculated from any n-rank tensor [2]. The normalisation of these metrics is achieved by using a scaling function, with parameters that can be changed to modify the contrast of the anisotropy maps. In this work we show how spectral decomposition [3] of the 4-rank generalised diffusion tensor can be used to characterise brain structure, including the definition of two metrics of anisotropy that do not depend on the arbitrary choice of a normalising function and its parameters.

## Theory:

Any 3D rank-4 tensor,  $\underline{D}$ , such that  $D_{ijmn}=D_{mni j}$ ,  $D_{ijmn}=D_{jimn}$  and  $D_{ijmn}=D_{ijnm}$  can be mapped into a 6D rank-2 tensor,  $\underline{S}$ , represented by a 6x6 symmetric matrix [3]. The rank-4 generalised diffusion tensor [1],  $\underline{D}$ , is a totally symmetric tensor, and therefore satisfies the required symmetry properties. This correspondence between the 4-rank tensor and the 6x6 matrix can be used to calculate the six eigenvalues ( $\sigma_1 > \sigma_2 > \sigma_3 > \sigma_4 > \sigma_5 > \sigma_6$ ) of the generalised diffusion tensor. The six eigenvalues for a 4-rank tensor corresponding to isotropic diffusion can be calculated analytically and are given by:  $\sigma_1 = 5 \langle D \rangle / 3$ ,  $\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 2 \langle D \rangle / 3$ , where  $\langle D \rangle$  is the mean diffusivity; the six eigenvalues corresponding to unidirectional diffusion are given by:  $\sigma_1 = (3 + \sqrt{11}) \langle D \rangle / 2$ ,  $\sigma_2 = \sigma_3 = \langle D \rangle$ ,  $\sigma_4 = \sigma_5 = 0$ ,  $\sigma_6 = (3 - \sqrt{11}) \langle D \rangle / 2$ . This eigenvalue decomposition of the 4-rank tensor can be used to define generalised fractional anisotropy (GFA) and generalised relative anisotropy (GRA):

$$GFA = \sqrt{\frac{\left(\sigma_1 - \frac{5 \langle D \rangle}{3}\right)^2 + \left(\sigma_2 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_3 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_4 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_5 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_6 - \frac{2 \langle D \rangle}{3}\right)^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2}}$$

$$GRA = \sqrt{\frac{\left(\sigma_1 - \frac{5 \langle D \rangle}{3}\right)^2 + \left(\sigma_2 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_3 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_4 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_5 - \frac{2 \langle D \rangle}{3}\right)^2 + \left(\sigma_6 - \frac{2 \langle D \rangle}{3}\right)^2}{\sqrt{5 \langle D \rangle^2}}}$$

Finally, the 3x3 eigentensors of the 4-rank generalised diffusion tensor can be used to construct "eigensurfaces" representing the projection  $P$  of  $\underline{D}$  along any direction

$$r: P(r) = \sum_{k=1}^6 \left( \sum_{i=1}^3 \sigma_k \gamma_i^k (r \cdot \underline{v}_i^k)^2 \right)^2, \text{ where } \gamma_i^k \text{ and } \underline{v}_i^k \text{ are the eigenvalues and the } 3 \times 1$$

eigenvalue vectors of each of the six 3x3 eigentensors.

## Methods:

Brain scans were obtained for a healthy volunteer using a Siemens 3T Trio imaging system. Diffusion weighted images were acquired along 145 unique sampling directions (b-value=2000 s/mm<sup>2</sup>, NEX=4), as well as nine b=0 images. The data was fitted to the generalised 4-rank diffusion tensor model and, after spectral decomposition, brain maps were obtained for the six eigenvalues, the mean eigenvalue  $\langle \sigma \rangle$ , GFA and GRA. The eigensurfaces given by  $P(r)$  were also computed for each voxel.

## Results:

Figure 1 shows the images obtained for the six eigenvalues,  $\langle \sigma \rangle$ , GFA and GRA. The eigenvalues have dimensions of diffusivity (mm<sup>2</sup>/s), and  $\langle \sigma \rangle$  shows a contrast similar to the one seen in mean diffusivity maps. Figure 2 shows the eigenvalue distribution across different regions of interest in the brain. For regions well characterised by one fibre population (such as the corpus callosum or the internal capsule) there is a significant difference between all eigenvalues, while for more isotropic regions (e.g., grey matter) the five smallest eigenvalues are very close to each other. Both GFA and GRA show a contrast between white and grey matter similar to what is obtained with other anisotropy metrics (e.g., fractional anisotropy). Figure 3 shows the eigensurfaces obtained for voxels in two different brain areas. For regions well characterised by one fibre populations the eigensurfaces have the shape of a peanut, while for regions of crossing fibres these eigensurfaces begin to exhibit a four-leaf type pattern.

**Conclusion:** The distribution of the six eigenvalues of the 4-rank diffusion tensor can be used to characterise different types of brain tissue. Two new metrics of anisotropy were also presented. Future work will investigate how the six eigenvalues, GFA and GRA may change in the presence of different pathologies.

**References:** [1] MRM 50: 955-965, 2003. [2] MRM 53: 866-876, 2005. [3] Signal Processing 87: 220-236, 2007.

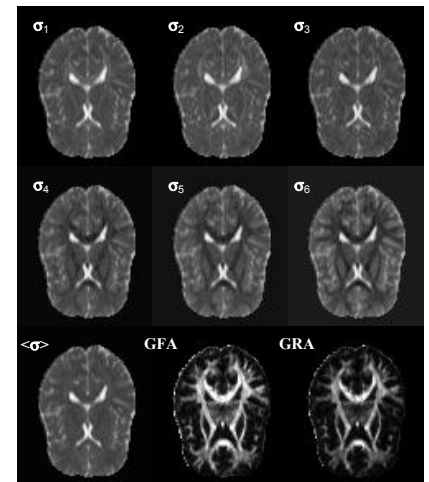


Figure 1 – Brain maps for the six eigenvalues,  $\langle \sigma \rangle$ , GFA and GRA.

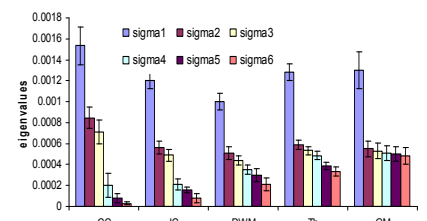


Figure 2 – Eigenvalue distribution in different brain regions.

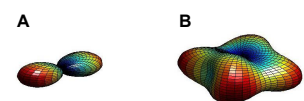


Figure 3 – Eigensurfaces for voxels in: A. the corpus callosum and B. intersection of radiations from the corpus callosum with the anterior part of the corona