Analysis of quadratic field distortions using the Fractional Fourier Transform

C. SING-LONG[1,2], V. PAROT[1,2], C. LIZAMA[3], S. URIBE[2,4], C. TEJOS[1,2], AND P. IRARRAZAVAL[1,2]
[1]DEPARTMENT OF ELECTRICAL ENGINEERING, PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE, SANTIAGO, CHILE, [2]BIOMEDICAL IMAGING CENTER, PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE, SANTIAGO, CHILE, [3]DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSIDAD DE SANTIAGO DE CHILE, SANTIAGO, CHILE, [4]DEPARTMENT OF RADIOLOGY, PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE, SANTIAGO, CHILE

INTRODUCTION: In Magnetic Resonance Imaging (MRI), the distortions produced by field inhomogeneities can be corrected with post processing techniques, e.g. linear correction and conjugate phase reconstruction methods [1][2]. However, these methods do not provide a theoretical framework to analyze the distortions. In this work, propose the Fractional Fourier Transform (FrFT) [3] as a way to study the distortions produced by quadratic field inhomogeneities. Concretely, we exploit the relation between the measured MR signal and the FrFT of the magnetization of an object under quadratic fields. We analyze some commonly used sequences to exemplify the usefulness of this method.

THEORETICAL BACKGROUND: For simplicity, we show the theory in one dimension. If $p(x) \approx p_0 + p_1 x + p_2 x^2$ is the field inhomogeneity (we assume that it can be well approximated by a quadratic function) and $k(t)$ is $\int_0^t G(s)ds$ as usual, the MR signal becomes:

$$s(t) = \int m(x) e^{-2i(k(t)x + p(x)t)} dx = e^{-2\pi i p_0 t} \int m(x) e^{i\pi(-2p_2 t x^2 - 2(k(t)+p_1 t)x)} dx$$

Using the change of variables $\cot\alpha(t) = -2p_2 t$ and $2\rho(t)\csc\alpha(t) = -2(k(t) + p_1 t)$ we can reformulate the signal equation as the FrFT of the underlying object:

$$s(t) = e^{-2\pi i p_0 t} \int m(x) e^{i\pi(-2p_2 t x^2 - 2(k(t)+p_1 x)t)} dx = e^{-2\pi i p_0 t} \int m(x) e^{i\pi(\cot\alpha(t) x^2 + 2\csc\alpha(t)\rho(t)x)} dx$$

$$= e^{-2\pi i p_0 t} e^{-i\pi\cot\alpha(t)\rho^2(t)} \sqrt{1 - i\cot\alpha(t)}\; F^{\alpha(t)}[m](\rho(t))$$

where $F^{\alpha}[m](\rho)$ represents the FrFT of fractional order α of the magnetization m measured at the fractional frequency ρ [3]. By writing $s'(t) = e^{2\pi i p_0 t} e^{i\pi\cot\alpha(t)\rho^2(t)} \sqrt{1 - i\cot\alpha(t)}^{-1} s(t)$ we obtain the relation $s'(t) = F^{\alpha(t)}[m](\rho(t))$. If $p_2 = 0$ then $\alpha(t) \equiv \pi/2$, $F^{\alpha}[m](\rho) \equiv F[m](\rho)$ and $s'(t) = s(t)$ and the classical relation between the MR signal and the Fourier transform of the magnetization is recovered. Consequently, the distortion produced by the quadratic field at time t is directly related to the order of the FrFT at the same instant. The quadratic inhomogeneity thus changes the frequency domain over which we perform the measurements, from the "pure" frequencies in the Fourier domain to "fractional frequencies" in intermediate domains.

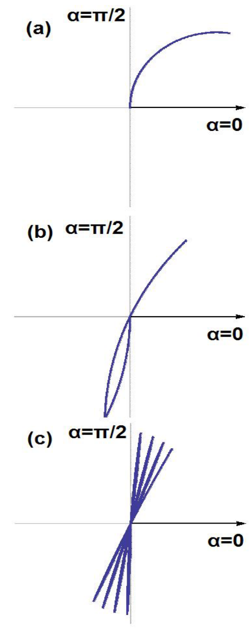

Figure 1

TRAJECTORY ANALYSIS: To visualize the distortions we introduce the notion of a $\rho \times \alpha$ plane or polar plane as a graphical analysis tool. These can also be represented by Cartesian coordinates $(x(t), y(t))$, with:

$$x(t) = \frac{2p_2 t(k(t) - p_1 t)}{1 + (2p_2 t)^2} \quad \text{and} \quad y(t) = -\frac{(k(t) - p_1 t)}{1 + (2p_2 t)^2}$$

Using this representation we study the "$\rho\alpha$–trajectories" described by $(\rho(t), \alpha(t))$ or $(x(t), y(t))$ for some commonly used k–space trajectories such as:

Radial readout: In this case we assume that $k(t) = G_0 t$ for $t \geq 0$. The $\rho\alpha$–trajectory is an arc of a circle centered at $a = \left(\frac{G_0 + p_1}{4p_2}, 0\right)$ with radius $r = \frac{G_0 + p_1}{4p_2}$ (Fig. 1a). We see that, as the ratio G_0/p_2 increases, so does the radius of this circle and the deviation from the vertical line becomes negligible for small t. This is consistent with the general knowledge that short readouts are less sensitive to inhomogeneities.

Standard DFT readout: We assume the gradient is formed by a negative lobule of duration t_0 followed by a positive one. In this case $k(t) = -G_0 t$ if $0 \leq t < t_0$ and $k(t) = G_0(t - 2t_0)$ if $t_0 \leq t$. The negative part of the sequence describes a $\rho\alpha$–trajectory which is an arc of a circle centered at $a_- = \left(\frac{G_0 + p_1}{4p_2}, -2G_0 t_0\right)$ with radius $r_- = \frac{1}{4p_2}\sqrt{(1 + 64 p_2^2 t_0^2)G_0^2 + 2p_1 G_0 + p_1^2}$, whereas the positive part describes an arc of a circle centered at $a_+ = \left(\frac{G_0 + p_1}{4p_2}, 0\right)$ with radius $r_+ = \frac{G_0 + p_1}{4p_2}$, as in the previous sequence (Fig. 1b).

EPI readout: In this case the sequence is composed by train of alternating square pulses of magnitude $\pm G_0$ and width $2t_0$. In this case the $\rho\alpha$–trajectory consists on arcs of circles with centers in $a_n = \left(\pm\frac{G_0 + p_1}{4p_2}, \mp\frac{G_0 t_0}{2} n\right)$ for $n = 0, 1, \ldots$ (Fig. 1c).

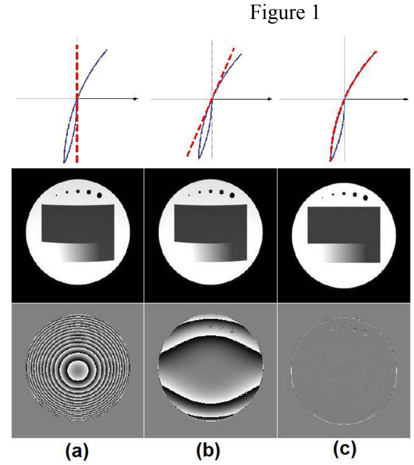

Figure 2

Spectroscopy: In this case no gradients are applied. Consequently, only the field inhomogeneity will contribute to the signal equation. The $\rho\alpha$–trajectory will be the same as in the radial sequence, namely, arcs of circles centered at $a = \left(\frac{p_1}{4p_2}, 0\right)$. As p_1 goes to zero, the trajectory collapses to the origin, which is interpreted as the continuous component of the signal for any fractional order α.

RECONSTRUCTION: The distorted samples can be seen as exact samples acquired on different positions in the $\rho\alpha$ domain. Figure 2 shows three different possible reconstructions. Column (a) shows the reconstruction obtained assuming that the samples are on the Fourier domain whereas in column (b) the samples are assumed to be in the constant order FrFT and in (c) they are assumed to be in the variable order FrFT. Note that FT reconstruction adds geometric and phase distortion. The reconstruction with constant order FrFT corrects most of the phase distortion but maintains the geometric distortions. Reconstruction with the variable order FrFT corrects both distortions.

CONCLUSIONS: In this work we have proposed a novel framework, based on the Fractional Fourier Transform to understand the distortions produced by quadratic field inhomogeneities. This framework allows interpreting these distortions as a domain change, from the "pure" frequencies in the Fourier domain to the "fractional frequencies" in the intermediary domain. This framework also allows quantifying the magnitude of these distortions. Consequently, this tool offers a way to study correction methods, both in image reconstruction and sequence development, for field inhomogeneities that can be well approximated by quadratic functions.

REFERENCES: [1] Irarrazaval *et al.*, MRM, 35(2), 1996, [2] Noll *et al.*, TMI, 24(3), 2005, [3] Ozaktas *et al.*, Wiley, 2001.