Comparative Evaluation of l_1 vs l_p Minimization Techniques for Compressed Sensing MRI

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Introduction

Compressed sensing (CS) has been shown to provide accurate reconstructions from highly undersampled data for certain types of MR acquisitions [1, 2]. This offers the promise of faster MR acquisitions, and further speed gains are possible when CS is used in conjunction with parallel acquisition schemes such as SENSE [3]. Several approaches have been recently proposed to reconstruct images from even fewer measurements than those required by standard ℓ_1-norm compressed sensing [2,4,5]. The purpose of this study was to test and compare standard ℓ_1-norm CS with two such approaches based on the ℓ_p quasi-norm [2,4,6] across different sampling pattern densities and parameterizations.

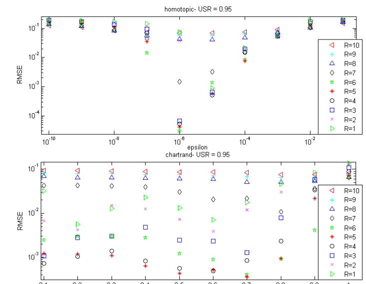

Figure 1. RMSE at USR=0.95 at different densities R. (above): H-ℓ_0 minimization, (below): ℓ_p minimization. The minimum RMSE found with H- ℓ_0 was on the order of 10^{-5} (at $\varepsilon=10^{-6}$); and 10^{-4} (at p=0.7) for ℓ_p

Theory

CS provides a method to reconstruct signals that are sampled well below the Nyquist limit, if the signal has a sparse representation under some transform Ψ, by selecting the solution with the sparsest representation in Ψ which still matches the limited observation set *f*. Such a search consists of solving the following equality-constrained ℓ_0-minimization problem:

$$\min_u \|\Psi u\|_0 \qquad \text{s.t.} \qquad \Phi u = \Phi f \qquad (1)$$

Direct addressing of the ℓ_0-minimization problem requires a combinatorial search of potential solutions which is intractable for most practical applications. In CS the ℓ_0-quasi-norm is replaced by the convex ℓ_1-norm, and (1) becomes a tractable minimization at the cost of requiring a modest degree of oversampling over the theoretical minimum required by the ℓ_o case. A natural question that arises is whether it is possible to find an alternative prior one can use to get closer to the ℓ_0- quasi-norm bound. Chartrand [4] proposed the use of the ℓ_p-norm, where 0<p<1. The corresponding l_p recovery problem becomes:

$$\min_u \|\Psi u\|_p^p \qquad \text{s.t.} \qquad \Phi u = \Phi f \qquad (2)$$

where $\|\Psi u\|_p^p \approx \left(|\Psi u|^2 + \varepsilon^2\right)^{p/2}$ and ε is a smoothing parameter. In [7] Chartrand and Yin proposed a continuation approach on ε. More recently, Trzasko and Manduca [2] proposed a generalization of the CS paradigm based on a homotopic approximation of the ℓ_0 quasi-norm, H-ℓ_0, which allows the use of different functionals. Here we consider H-ℓ_0 minimization using the same ℓ_p quasi-norm, with fixed ε but effectively doing continuation on *p*. Since both of these approaches involve non-convex ℓ_p minimization, there is no theoretical guarantee of convergence to the optimal solution as there is in the ℓ_1 case.

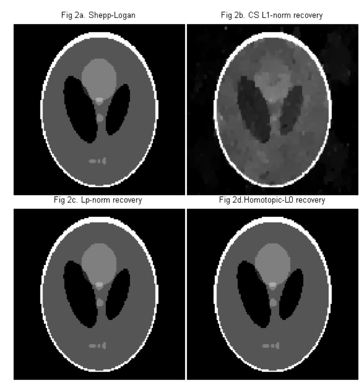

Figure 2. Comparison of image recovery using USR=0.95, R=6. (a) Original phantom (b) ℓ_1-norm CS recovery (c) ℓ_p-quasi-norm recovery using p=0.7 (d) H-ℓ_0 recovery using $\varepsilon=10^{-5}$.

Methods

We implemented the ℓ_1, ℓ_p and H- ℓ_0 algorithms using the spatial gradient as the sparsifying transform and tested them using a 128x128 Shepp-Logan phantom with 11 different undersampling rates USR = 0.75, 0.80, 0.85, 0.88, 0.90, 0.92, 0.93, 0.94, 0.95, 0.96, 0.97 and 0.98. Noting the use of variable-density k-space sampling strategies in [1] and [2], we also tested 10 different Gaussian k-space sampling parameterizations (standard deviation = N/R, for R = 1:10). For the ℓ_p algorithm we studied 10 parameterizations from 0.1<*p*<1.0. The H-ℓ_0 algorithm was tested with ε between 10^{-10} and 10^{-1}. We repeated each experiment 10 times with different random sampling patterns, performing 22,000 simulations in all. Reconstructions were assessed by calculating the root minimum, mean and maximum squared errors (RMNSE, RMSE, RMXSE) per pixel.

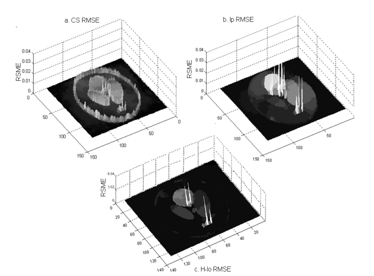

Figure 3. All images R =5, RMSE ~10^{-3}, right before failure to recover. (a) Standard CS, USR=0.92 (b) ℓ_p-quasi-norm USR=0.96, p=0.4 (c) H-ℓ_0 recovery with USR=0.96, $\varepsilon=10^{-5}$.

Conclusions

The simulations confirmed that it is possible to substantially reduce the number of samples required for exact reconstruction when using the ℓ_p-norm for values p<1. Standard ℓ_1-norm CS broke down at USR=0.93-0.94, whereas both the ℓ_p and H-ℓ_0 approaches could achieve accurate reconstructions for some R at USR=0.96-0.97. Despite the lack of theoretical guarantees of convergence, both ℓ_p-norm approaches always outperformed standard ℓ_1-norm CS. The simulation results suggest that the H-ℓ_0 approach works well under a broader range of sampling densities R, but it is sensitive to the choice of ε, while the ℓ_p algorithm is fairly insensitive to the choice of *p*, but is somewhat sensitive to the choice of R. At high USRs ~0.95, our simulations agree with previous observations that the ℓ_p approach does not provides better results as *p* goes below 0.5 [7], although this was not true at higher sampling rates below USR=0.92. When studying the location of the pixel errors at parameterizations at which each method was beginning to fail, we consistently observed that for both non-convex ℓ_p minimizations the error was concentrated in small, low contrast objects, while the ℓ_1-norm CS error was spread more widely across the image, and preferentially at high contrast borders and regions (see Figure 3). Further study investigating the properties of these approaches is required.

References

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