Evaluation of continuous approximation functions for the l_0-norm for Compressed Sensing

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Introduction: Compressed Sensing (CS) ([1], [2], [3], [4]) allows reconstructing a signal, if it can be represented sparsely in a suitable basis [4], from only a portion of its Fourier coefficients. It was first used by Lustig et al. [5] in MRI, and it has become popular for speeding up the acquisition process. Initially, CS was introduced as an l_0-norm minimization [1] which is in practice unsolvable since it is NP-hard. However if the number of acquired samples is increased, this l_0-norm minimization is equivalent to an l_1-norm minimization and therefore faster to compute [1], [2]. It was recently shown in [6] that the CS problem can be solved with less acquired samples than those needed for the l_1-norm problem, by minimizing iteratively continuous approximations of the l_0-norm. There are a handful continuous functions that can be used to approximate an l_0-norm and reconstruct sparse signals [7], but their reconstruction error and convergence properties vary significantly. In this paper we evaluate the performance of four approximation functions using a fixed-point CS solver.

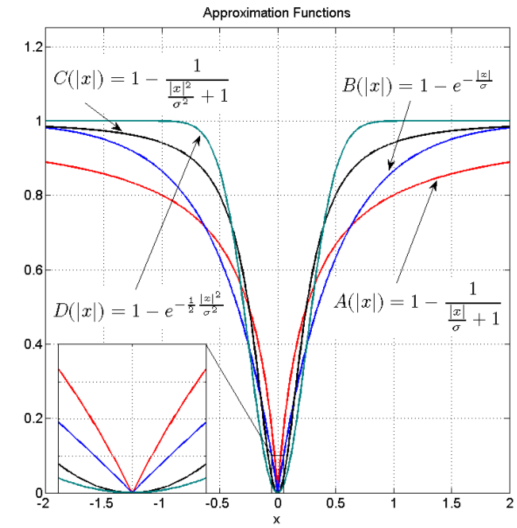

Figure 1

Methods: We used four different approximation functions (A, B, C, and D defined in Fig.1). Functions A and B have a discontinuous derivative in zero, whereas C and D have a continuous one. Also, A and C are rational functions whereas B and D are exponentials. To evaluate the convergence performance of these functions, we generated 10 vectors of size $N = 96$, with integer intensities randomly chosen from a uniform distribution $\{0 - 255\}$. With each vector we constructed 24 sparse signals with different support size S by choosing randomly $1, 5, \ldots, 89, 93$ data points and changing the rest to zero. Sparse signals were Fourier transformed and undersampled at random locations with 24 different undersampling rates (the number of sampled coefficients were $M = 2, 6, \ldots, 90, 94$), so that the total reconstruction experiments were $10 \times 24 \times 24 = 5760$. We implemented a reconstruction scheme similar to that used in [6] and [7] with a fixed-point optimization algorithm.

Let f and g be the original and reconstructed vector respectively. We compared the performance by considering as metrics the mean and standard deviation of the reconstruction error (defined as $\sqrt{\sum_n |f_n - g_n|^2} / \sqrt{\sum_n |f_n|^2}$), the total number of iterations and the number of non-zero values of the reconstructed signal (defined as $1 - |supp\{g\}|/|supp\{f\}|$). We also considered three regions of interest (see Fig.2), namely Region I ($M \geq 2S$), Region II ($S \leq M < 2S$) and Region III ($M < S$). These three regions represent the relative sampling rates where we know that an exact reconstruction can be found (Region I) as demonstrated in [1] for N prime; where it is impossible to achieve an exact reconstruction (Region III) as can be easily demonstrated by a degrees of freedom argument; and where there is no guarantee for any of the previous results (Region II).

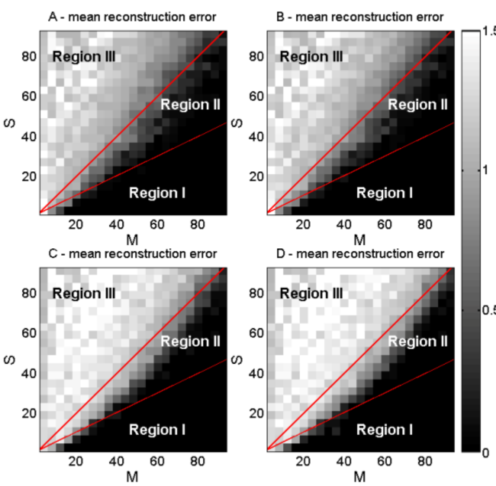

Figure 2

Results: As can be seen in Table 1, in Region I all four families reconstructed the signals with minimal error. However, functions with continuous derivative produced in average 30% and 50% more error in Regions II and III respectively. Additionally, we observed that functions with discontinuous derivatives needed around 88% less number of iterations than those with continuous derivatives to converge in Region I. In Regions II and III the number of required iterations needed for A, C and D was comparable. We also verified that, with exception of function D, all functions produced comparable errors when estimating the number of non-zero components of the original vector. In particular, this error was proportional to the ratio between undersampling rate and the number of non-zero components of the original vector. As an example, in Fig.2 we see how function A gradually increments the reconstruction error as we move from Region I to Region III. In contrast, function D sharply increased its reconstruction error. In summary, we conclude that approximating by function A is the best choice: it produces the smallest reconstruction errors in all three regions, while maintaining a reasonable small number of iterations in all regions.

Table 1		Rec. Error (std dev)	Iterations (std dev)	Non-zero el. (std dev)
I	A	0.009 (0.102)	164 (1729)	0.006 (0.078)
	B	0.010 (0.106)	254 (2136)	0.013 (0.288)
	C	0.011 (0.106)	1689 (1509)	0.016 (0.192)
	D	0.015 (0117)	1918 (1628)	0.084 (1.740)
II	A	0.199 (0.343)	7855 (12401)	0.053 (0.120)
	B	0.203 (0.350)	13770 (20615)	0.050 (0.117)
	C	0.313 (0.457)	7833 (6091)	0.043 (0.094)
	D	0.353 (0.490)	7491 (4424)	0.173 (0.427)
III	A	1.083 (0.473)	8293 (8304)	0.509 (0.285)
	B	1.123 (0.479)	15913 (13692)	0.509 (0.284)
	C	1.310 (0.472)	7624 (4644)	0.508 (0.325)
	D	1.352 (0.477)	6334 (3350)	0.400 (0.419)

Conclusions: We have studied four possible families of functions that could be used to approximate the l_0-norm and reconstruct undersampled images using the CS theory in an iterative fixed-point optimization. We analyzed the performance of these families for three conditions: (1) the number of samples is at least the double of the support size of the signal; (2) the number of samples is between the support size and its double; and (3) the number of samples is less than the support size. From this analysis, we conclude that function A should be the choice to approximate the l_0-norm, leading to fast and accurate reconstructions using CS. Additionally we conclude that there are several instances in which the algorithm can reconstruct the signal in the uncertain middle condition. Finally, we notice that functions with discontinuous derivatives reconstruct well the signals and may have better convergence properties, suggesting that the results in [7] may be extended to these functions.

[1] Candès et al, IEEE TIT, 52(2), pp. 489 – 509, [2] Candès et al, IEEE TIT, 51(12), pp.4203 – 4215, [3] Donoho, IEEE TIT, 52(4), pp.1289 – 1306, [4] Candès et al, Inv. Prob., 23(3), pp.969 – 985, [5] Lustig et al, MRM, 58(6), pp.1182 – 1195, [6] Trzasko et al, IEEE TMI, *In Press*, [7] Mohimani et al, IEEE TSP, *In Press*.