Recoverability Bounds for Parallel Compressive Sensing MRI

J. D. Trzasko[1], and A. Manduca[1]

[1]Center for Advanced Imaging Research, Mayo Clinic, Rochester, MN, United States

Introduction The marriage of Compressive Sensing (CS) [1,2,3] reconstruction techniques with multi-coil methods such as SENSE [4] offers a promising means for significantly accelerating MR image acquisition [5-11]. While there has been much theoretical analysis on the individual performance of these two methods, to date there has been only minimal investigation into the performance of the hybrid approach. In this work, we investigate the theoretical performance of the multi-coil CS framework and derive a relationship between signal recoverability and certain properties of the coil sensitivity profile set.

Theory Let Φ denote a K-point discrete Fourier transform (DFT) matrix and Γ_c be a diagonal matrix describing the c^{th} coil sensitivity profile. The undersampled parallel MR image acquisition process for an N-point image, f, is then typically defined by [4]

$$Ef = g = [g_1 \ldots g_c]^T, \quad \text{where } E = [\Phi\Gamma_1 \ldots \Phi\Gamma_c]^T. \tag{1}$$

Now suppose the signal of interest, f, is (intrinsically) sparse, i.e. $\|f\|_0 = S << N$. If

$$S < \tfrac{1}{2}\left(1 + [\mu(E)]^{-1}\right), \tag{2}$$

where $\mu(E)$ is the mutual coherence (to be defined) of the sensitivity matrix E, then f can be exactly recovered using either the convex Basis Pursuit (ℓ_1-minimization) or greedy Matching Pursuit algorithms [12,13]. While this bound is certainly not sharp, it is nonetheless clearly desirable to have as small a $\mu(E)$ as possible due to its inverse relationship with the number of recoverable coefficients. Empirical optimization of sampling matrix coherence for single-sensor systems has been previously considered [3,14,15]. When E is defined as in (1) for the case of parallel MRI, the coherence term can be defined and then algebraically separated as

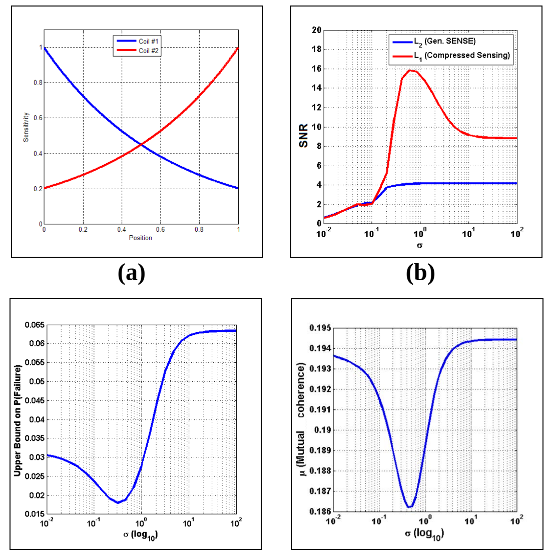
$$\mu(E) = \max_{p \neq q}\left\{\frac{\left|\langle E_p, E_q\rangle\right|}{\|E_p\|_2 \cdot \|E_q\|_2}\right\} = \max_{p \neq q}\left\{\frac{\left|\langle \Phi_p, \Phi_q\rangle\right|}{\|\Phi_p\|_2 \cdot \|\Phi_q\|_2} \cdot Q_{q,p}\right\} \tag{3}$$

where A_n denotes the n^{th} column of A and the coil sensitivity profile set measure is defined as

$$Q_{q,p} = \frac{\left|\sum_{c=1}^{C}\left[diag\left\{\Gamma_c^H\right\}\right](q)\cdot\left[diag\left\{\Gamma_c\right\}\right](p)\right|}{\sqrt{\sum_{c=1}^{C}\left|\left[diag\left\{\Gamma_c\right\}\right](q)\right|^2}\cdot\sqrt{\sum_{c=1}^{C}\left|\left[diag\left\{\Gamma_c\right\}\right](p)\right|^2}} \tag{4}$$

Note that $Q_{q,p}$ is essentially the coherence of the sensitivity profiles; however, unlike traditional coherence which is a spatial measure, this metric is across the channels at a fixed spatial position. When the K constituent frequencies (rows) of Φ are randomly sampled, one generally considers minimizing the probability that $\mu(E)$ exceeds a desired threshold τ rather than determining instance optimality. Using the symmetric variant of McDiarmid's bounded difference inequality [16], it can be shown that

$$P(\mu(E) \geq \tau) \leq 2 \cdot \sum_{p \neq q} \exp\left\{-\frac{K}{2}\left(\frac{\tau}{Q_{q,p}}\right)^2\right\} \tag{5}$$



(a) **(b)** **(c)** **(d)**

Figure 1. *(a) Two simulated 1D coil profiles exhibiting exponential decay (σ=0.6). (b) SNR of the reconstructed Shepp-Logan phantom for different values of σ using both SENSE and the hybrid CS+SENSE methods. Note that the clear optimality peak in the CS result. (c) Plot of the upper bound derived in (5) showing that the probability that μ(E) exceeds some threshold is minimal at roughly the same configuration yielding optimal SNR in (b). (d) Monte Carlo simulation of μ(E) for 1000 random Fourier trials at each value of σ. Note that the coil configuration yielding minimal μ(E) corresponds with the minimum of the bound in (c) and again the optimality peak in (b).*

As a consequence, coil sensitivities can, in theory, be optimized for CS-type reconstructions by minimizing the upper bound in (5), ensuring that the mutual coherence of the random measurement ensemble is below the prescribed threshold with high probability. Similar bounds were also derived for the Gaussian and Bernoulli/Rademacher measurement ensembles commonly employed in the CS literature.

Example Consider the simple example with two coils (C=2) exhibiting exponential falloff with a variable rate σ (e.g. Fig. 1a). As $\sigma \rightarrow \infty$, the profiles tend towards spatial uniformity. Fig. 1b shows the SNR reconstruction results for the noise-free Shepp-Logan phantom with ~6000 random Fourier coefficients and various σ's using SENSE and ℓ_1-minimization (with Haar wavelets). Whereas SENSE is fairly invariant to coil falloff rate for most larger (and thus numerically stable) values of σ, the hybrid CS+SENSE method is distinctly optimal near σ=1. For the same coil profile set used in Fig. 1b, Fig. 1c shows (5) with K = 128 and τ=0.5. In this example, note that there is a clear minimum in the probability that the mutual coherence exceeds the prescribed threshold, and note that this minimum occurs proximal to coil parameterization yielding maximal SNR in Fig 1b. Moreover, the Monte Carlo simulation of $\mu(E)$ (1000 randomly undersampled Fourier matrices for each tested σ) shown in Fig. 1d further confirms the ability of (5) to accurately predict the optimal coil configuration.

Conclusion In this work we derived a new relation for signal recoverability via Compressive Sensing when it is applied to parallel MR image acquisition. Both theoretical and empirical evidence suggest the potential of this bound model for future work in developing optimal coil configurations that will provide optimal image quality when employed in a CS reconstruction framework.

References

[1] E. Candès et al., IEEE TIT 52(2), 2006. [2] D. Donoho, IEEE TIT 52(4), 2006. [3] M. Lustig et al., MRM 58(6), 2007. [4] K. Pruessmann et al. MRM 46(4), 2001. [5] K. Block et al., MRM 57(6), 2007. [6] B. Liu et al. Proc. ISMRM 2008, #13. [7] C. Zhao et al. Proc .ISMRM 2008, #1478. [8] B. Wu et al. Proc. ISMRM 2008, #1480. [9] L. Marinelli et al. Proc ISMRM 2008, #1484. [10] K. King. Proc ISMRM 2008, #1488. [11] B. Liu et al. Proc ISMRM 2008, #3154. [12] D. Donoho and M. Elad. PNAS 100, 2003. [13] R. Gribonval and M. Nielsen. IEEE TIT 49(12), 2003. [14] M. Elad. IEEE TSP 55(12), 2007. [15] Y. Kim and K. Nayak. Proc. ISMRM 2008, #422. [16] C. McDiarmid. Surveys in Combinatorics, 1989.