# The Correct Method for Numerical Calculation of Induced Electric Fields for Rotational Movements in Static Magnetic Fields

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**Introduction:** It is well known that electric fields are induced in the human body due to changes in magnetic field with time. High rates-of-change of magnetic field in the 0.1 Hz – 10 kHz range can cause peripheral nerve stimulation (PNS), magnetophosphenes, vertigo, and metallic taste [1, 2]. Analytic calculations and numerical modelling have previously been used to estimate the magnitude of the electric fields induced in the body during magnetic field gradient switching and natural movements in the environment of scanner magnets[3-5]. The aim of the work described here is to highlight a potential error in the approach that has been used in a number of previous investigations involving calculating of the currents induced in the body by rotation (e.g. bending, leaning or twisting) in magnetic fields. Errors arise from calculation of the induced electric field in terms of the rate of change of the vector potential rather than $\mathbf{v}\times\mathbf{B}$ and from neglecting the presence of the induced space charge. We show that use of the incorrect formulation can give rise to large errors in the calculated electric fields and current densities.

**Theory:** The electric field , $\mathbf{E}'$ , induced in a body moving at velocity, **v**, in a magnetic field, **B**, is given by Faraday's Law as $\nabla\times\mathbf{E}' = -\partial\mathbf{B}/\partial t + \nabla\times(\mathbf{v}\times\mathbf{B})$ [Eqn. 1] where the electric field is calculated in the moving frame of reference of the object and the magnetic field and its rate of change are defined in the stationary, laboratory frame [6, 7]. The $\mathbf{v}\times\mathbf{B}$ term arises from considering the Lorentz force on charges in the object as it moves in a magnetic field.

When calculating the induced electric field it is convenient to use the scalar-potential finite-difference (SPFD) method (or similar) and to define the magnetic field in terms of the vector potential, **A**, (i.e. $\mathbf{B}=\nabla\times\mathbf{A}$) so that the electric field may be written as, $\mathbf{E} = -\partial\mathbf{A}/\partial t + \mathbf{v}\times\mathbf{B} - \nabla V$ [Eqn. 2] where *V* is a scalar potential. This scalar potential arises due to surface charges which form in order to satisfy continuity of current at boundaries. In the situation where the object is stationary and the magnetic field varies in time, the electric field may be correctly written as, $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla V$ [Eqn. 3]. However whilst this expression is valid for stationary systems, errors occur when it is applied to a moving system and the rate of change of **A** is calculated as the difference in **A** measured at two points separated by an infinitesimal distance in the direction of local motion divided by the time taken to travel between these points i.e. setting $\mathbf{E} = -(\mathbf{v}\cdot\nabla)\mathbf{A} - \nabla V$ [Eqn. 4]. To examine the origin of this error, let the motion of the body (assumed to be rigid) be defined as a sum of a translation at velocity $\mathbf{v_0}$ and a rotation with angular velocity $\boldsymbol{\Omega}$, such that $\mathbf{v}(\mathbf{r}) = \mathbf{v}_0 + \boldsymbol{\Omega}\times\mathbf{r}$ . In this situation $\mathbf{v}\times\mathbf{B} = -(\mathbf{v}\cdot\nabla)\mathbf{A} + \nabla(\mathbf{v}\cdot\mathbf{A}) + \boldsymbol{\Omega}\times\mathbf{A}$ [Eqn. 5] which makes clear that calculating the induced electric field in terms of $-(\mathbf{v}\cdot\nabla)\mathbf{A}$ may give a very different result than use of $\mathbf{v}\times\mathbf{B}$. Since $\nabla\times\nabla(\mathbf{v}\cdot\mathbf{A}) = 0$ discrepancies occur whenever $\nabla\times(\boldsymbol{\Omega}\times\mathbf{A})$ is non-zero. Thus for translational motion ($\boldsymbol{\Omega}$ = 0) use of either $-(\mathbf{v}\cdot\nabla)\mathbf{A}$ or $\mathbf{v}\times\mathbf{B}$ will yield the correct electric field, but for rotational motion it is necessary to use $\mathbf{v}\times\mathbf{B}$ as the driving term unless $(\boldsymbol{\Omega}\times\mathbf{A})$ is irrotational. In addition, by computing the divergence of **E'** we can conclude that in the case of rotational motion a space charge term $\rho = \varepsilon_0\nabla\cdot\mathbf{E}' = -\varepsilon_0\nabla\cdot(\mathbf{v}\times\mathbf{B}) = -2\varepsilon_0\boldsymbol{\Omega}\cdot\mathbf{B}$ may be induced. It can be noted that this induced charge density is independent of the conductivity and is generated if the system rotates in a magnetic field that is not perpendicular to the axis of rotation. This space charge gives rise to a conservative electric field which generally opposes the induced electric field. In some cases these fields completely cancel out so that no currents flow in the object. It is worth noting that the permittivity of free space is seen in this expression rather than the permittivity of the object because the space charge depends on the free charges within a conductor and not on the polarization charges in a dielectric. In the SPFD method this additional charge can be modelled using an additional scalar potential, *V'*, such that $\nabla^2 V' = 2\boldsymbol{\Omega}\cdot\mathbf{B}$. Table 1 summarises the expected characteristics of the charge and current densities induced by rotational and translational movements in a static magnetic field when the correct formalism (Eqn.1) is used. The conditions needed for current to flow are also noted in the Table.

| MOVEMENT \ FIELD | UNIFORM $\mathbf{B} \neq \mathbf{B}(\mathbf{r})$ | NON UNIFORM $\mathbf{B} = \mathbf{B}(\mathbf{r})$ |
|---|---|---|
| TRANSLATION | $\rho = 0$ | $\rho = 0$ |
| | $\mathbf{J} = 0$ | $\mathbf{J} \neq 0$ if $(\mathbf{v}\cdot\nabla)\mathbf{B} \neq 0$ |
| ROTATION | $\rho = -2\varepsilon_0\Omega\cdot\mathbf{B}$ | $\rho = -2\varepsilon_0\Omega\cdot\mathbf{B}$ |
| | $\mathbf{J} = 0$ if $\Omega \parallel \mathbf{B}$ | $\mathbf{J} = 0$ if $\nabla\times(\mathbf{v}\times\mathbf{B}) = 0$ |

**Table 1:** Current and charge induced for movements in magnetic fields.

**Methods and Results:** To demonstrate the effect of neglecting the $(\boldsymbol{\Omega}\times\mathbf{A})$ term, the induced electric field has been calculated for three specific examples, spanning analytic and numerical methods:

1) A uniform conducting sphere rotates at angular velocity, Ω, about the x-axis in a uniform magnetic field, $B_0\mathbf{k}$. In this case no space charge is produced and we find that the electric field calculated using $\mathbf{v}\times\mathbf{B}$ is $0.5\Omega B_0(-z\mathbf{i} + x\mathbf{k})$, while using $-(\mathbf{v}\cdot\nabla)\mathbf{A}$ gives the same spatial form of electric field, but with half the magnitude.

2) A uniform conductive sphere of radius, a , rotates about the z-axis in a longitudinal magnetic field gradient, of strength, G . In this case a space charge is generated, but there should be no induced electric field as $\nabla\times(\mathbf{v}\times\mathbf{B}) = 0$ , and this is confirmed by calculation based on the use of $\mathbf{v}\times\mathbf{B}$ which yields **E** =0 throughout the sphere. However, as $\nabla\times(\boldsymbol{\Omega}\times\mathbf{A}) \neq 0$ we find that using $-(\mathbf{v}\cdot\nabla)\mathbf{A}$ incorrectly predicts a finite spatially varying electric field whose magnitude is of order $\Omega G a^2$.

3) For a more complex geometry a boundary element method (BEM) numerical calculation (incorporating a space charge term) was used to calculate the current induced in a head model due to rotation. In this case only 3 domains were modelled (brain, skull and scalp). A non-uniform magnetic field perpendicular to the y-axis has been simulated such that $\mathbf{B} = (-x\hat{\mathbf{i}} + z\hat{\mathbf{k}})$ and the head rotates at 1 rad $s^{-1}$ about the x-axis. Two simulations were performed, one using $\mathbf{v}\times\mathbf{B}$ term and the other $-(\mathbf{v}\cdot\nabla)\mathbf{A}$ as the electric field driving terms. Figure 1 shows the magnitudes of current density calculated using the two different driving terms. The overall scaling of the map in Fig 1a is double that of Fig 1b and specific locations can be found where the current densities are appreciably different in both magnitude and direction in the two maps.

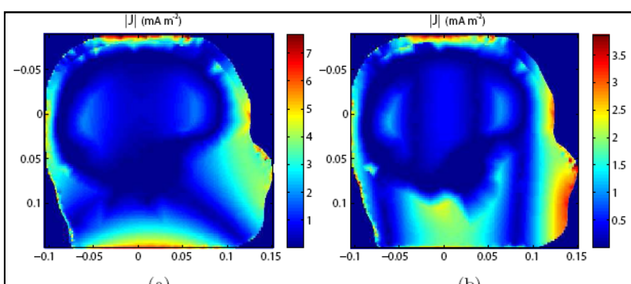


**Figure 1:** Modulus of the induced current density in the central xz-plane for the scalp-skull-brain system rotating in a uniform static magnetic field $\mathbf{B} = (-x\hat{\mathbf{i}} + z\hat{\mathbf{k}})$, computed using (a) **v** x **B** and (b) $-(\mathbf{v}\cdot\nabla)\mathbf{A}$

**Conclusions:** The use of the vector potential in electromagnetic simulations of motion may lead to incorrect values of current density being calculated if the term **Ω** x **A** is neglected. This error may be remedied by always using **v** x **B** as the driving term for all motions in magnetic fields. It is essential to remember that motion of a conductive body in a magnetic field is not equivalent to a stationary body in a moving magnetic field.

**References:** 1.) Irnich, W., *et al*., Mag. Res. Med. 1995. **33(5)** 619-623. 2.) Schenck, J.F., *et al*., Med. Phys. 1992. **19(4)** 1089-1098. 3.) Bencsik, M., *et al*., Phys. Med. Biol. 2002. **47(4)** 557-576. 4.) Liu, F., *et al*., Concepts In Mag. Res. 2002. **15(1)** 26-36. 5.) Liu, F., *et al*., J. Mag. Res. 2003. **161(1)** 99-107. 6.) Lorrain, P., Eur. J. Phys. 1990. **11** 94-98. 7.) Redzic, D.V., Eur. J. Phys. 2007. **28(5)** N7-10.