The Mathematics of HYPR

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Introduction

A promising new method for accelerated imaging, HighlY constrained backPRojection (HYPR) [1] uses the spatial information of a time averaged image (composite) to improve images at individual time frames. The original HYPR algorithm [1] was derived heuristically based on the idea that the location of the object was known using the composite image and that the role of the data from each time frame was to extract the intensity profile at that time. That perspective led to a practical understanding of the algorithm that has been generalized and modified, as in the HW-HYPR method [2] which modified HYPR so that the method converges to the composite when the time frame uses all projections. The HYPR approach has also been extended to an iterative method (I-HYPR) [3] in which that version of the HYPR algorithm was identified by the authors as the expectation maximization (EM) algorithm. The objective of this work is to clarify the mathematical understanding of the original HYPR algorithm and HW-HYPR as the first step of iterative methods used in statistical image reconstruction in other imaging modalities which enforce a positivity constraint (if the initial guess is non-negative, the resulting image will be non-negative also) with an initial condition that encodes the spatial location of the object being reconstructed (i.e. the composite).

Mathematical Background

The two main mathematical ideas needed to explain HYPR are maximum likelihood estimation (MLE) and fixed point iteration (FPI) [4]. In MLE the parameter being estimated is chosen such that the probability that the data was generated by that parameter is maximized. Let $\mathbf{g} = \mathrm{H}\mathbf{f} + \mathbf{n}$, where $\mathbf{g}$ is the data vector, H is the imaging operator (maps object to projections in a radial acquisition), $\mathbf{f}$ is the object and $\mathbf{n}$ is noise. An MLE estimate of $\mathbf{f}$ maximizes $P(\mathbf{g}\,|\,\mathbf{f})$ as a function of $\mathbf{f}$, i.e. the probability of observing the acquired data given the object. Finding the maximum requires solving an equation with large matrices. Iterative methods using multiplicative updates is one way of solving large systems of equations: $\mathbf{b} = \mathrm{A}\mathbf{x}$ can be solved by:

$$\mathbf{x}_k^{n+1} = \mathbf{x}_k^n \left(\frac{\mathbf{b}}{\mathrm{A}\mathbf{x}^n} \right)_k$$

where n is the iteration number, and k is the pixel index. If the iteration converges to a fixed point, then the algorithm reaches a solution to the original equation, hence the term fixed point iteration (FPI). The part in parenthesis is a multiplicative update to the previous guess ($\mathbf{x}_k^n$).

Results

An FPI for estimating the object assuming Poisson noise leads to a well-known algorithm used for limited angle tomography in nuclear medicine, Maximum Likelihood Expectation Maximization (MLEM) [4,5]:

$$f_k^{n+1} = f_k^n \frac{1}{s_k} \left(\mathrm{H}^t \left[\frac{\mathbf{g}}{\mathrm{H}\mathbf{f}^n} \right] \right)_k$$

where s_k is a scaling constant that depends on the pixel (in a radial acquisition, the backprojection of a set of projections with a value of one in every detector pixel), and H^t is the transpose of the imaging operator (in a radial acquisition, the backprojection operation). O'Halloran et al. [3] identify I-HYPR as the EM algorithm with the HYPR image as the first step, but the formulation of HYPR in [3] is not equivalent to the formulation in the original HYPR [1]. Here we show that the original HYPR algorithm itself is approximately the first step of the MLEM algorithm with the composite image as the initial guess under the approximation that the multiplicative updates are the same:

$$\frac{1}{N_p} \left[\sum_{i=1}^{N_p} \frac{\mathrm{H}_i^t(\mathbf{g}_i)}{\mathrm{H}_i^t([\mathrm{H}\mathbf{f}]_i)} \right]_k = \frac{1}{s_k} \left[\mathrm{H}^t \left(\frac{\mathbf{g}}{\mathrm{H}\mathbf{f}} \right) \right]_k$$

where N_p is the number of projections per time frame and the index i refers to the individual projections, up to a normalization constant this equation implies that the ratio of backprojections is the backprojection of the ratio, which is an approximation whose accuracy is object dependent. Figure 1 shows the difference in error for a noise-free time-independent simulation of a disk using 4 projections per time frame between the first step of the MLEM algorithm, the HYPR reconstruction and the HW-HYPR reconstruction. Figure 2 shows images of the multiplicative updates (shown in the above equation) for the simulation in Figure 1.

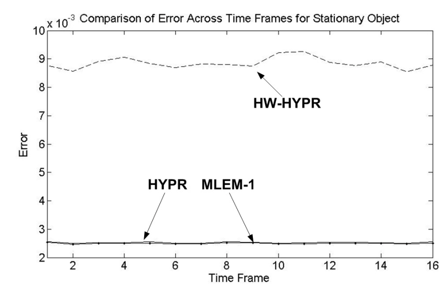

Fig. 1 Simulation of a stationary object shows HYPR and the first step of MLEM (using composite as the initial guess) are the same but HW-HYPR is different. The lines for HYPR and MLEM-1 overlap.

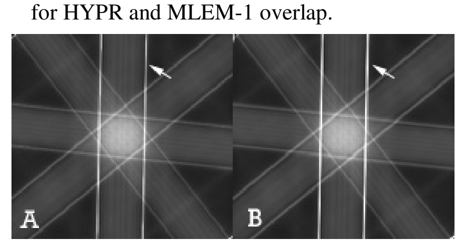

Fig.2 Comparison of multiplicative update for HYPR (A) and first step of MLEM (B). There are subtle differences outside of the object support as shown by the arrow.

HW-HYPR is a variant of the HYPR algorithm which was originally derived to have the composite image as the limiting reconstruction when all projections are used in a time frame [2], and can also be derived in the framework of an MLE obtained by an FPI, but instead of using Poisson noise, it uses the Gaussian noise model and applies a multiplicative update to solve the normal equation: $\mathrm{H}^t\mathbf{g} = \mathrm{H}^t\mathrm{H}\mathbf{f}$ leading to:

$$f_k^{n+1} = f_k^n \left(\frac{\mathrm{H}^t\mathbf{g}}{\mathrm{H}^t\mathrm{H}\mathbf{f}^n} \right)_k$$

This shows that HW-HYPR is the first step of the Multiplicative Arithmetic Reconstruction Technique (MART) [6] applied to the normal equations using the composite as the initial guess.

Discussion

The connection of HYPR with MLEM (for Poisson noise) and HW-HYPR with MART (for Gaussian noise) helps to understand these new algorithms within the context of algorithms studied in other imaging modalities. The success of these new algorithms can be understood in terms of the prior information they utilize, i.e. the compact support given by the composite image and the positivity constraint for the radial acquisitions. The iterative reconstruction beyond the first step (HYPR does not iterate) further enforces agreement between the object and the data leading to the natural trade-off between data agreement and noise amplification. By understanding these methods as first step estimates of an MLE we can also better understand their behavior with respect to different noise sources. In applications like time-resolved angiography, it may be more important to impose prior information than to use the correct noise model. This could explain the success of HYPR which incorporates a positivity constraint and knowledge of the spatial location of the object being reconstructed.

References

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