Pearson set of distributions as improved signal model for diffusion kurtosis imaging

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Introduction

Diffusion kurtosis imaging is a new method to estimate the non gaussianity of the diffusion process with diffusion weighted MR images (DWIs) [1]. The model of the DWIs suggested in [1] was based on the Taylor series expansion of the logarithm of the DW magnitude. Unfortunately, this Taylor approximation is only valid for small b-values and leads to diverging predicted DW magnitudes when the kurtosis is positive and the b-value is large, see figure 1.

Methods

In this work we propose a new model for the DW magnitude, which also includes the kurtosis of the diffusion process. This new model is based on the Pearson set of statistical distributions [2], and, since it is based on true probability density functions (pdf's), it provides more realistic predictions of the DW magnitude. The Pearson set of distributions is a set of pdf's, specifically designed to allow free adjustability of the mean, variance, skewness and kurtosis of the distribution. Since the diffusion process is symmetric, only the subset of the pdf's with zero mean and skewness is needed. For negative excess kurtosis, the Pearson type II distribution and for positive excess kurtosis, the Pearson type VII distribution should be used, respectively,

$$P_{II}(x) = \frac{\Gamma\left(\frac{3}{2}-m\right)}{\sqrt{\pi}\,\Gamma(1-m)s}\left[1-\left(\frac{x}{s}\right)^2\right]^{-m} \quad \forall \quad x < s\,, \qquad P_{VII}(x) = \frac{1}{\alpha B\left(m-\frac{1}{2},\frac{1}{2}\right)}\left[1+\left(\frac{x}{\alpha}\right)^2\right]^{-m}$$

with

$$\alpha = \sigma\sqrt{2m-3}, \quad s = \sigma\sqrt{3-2m}, \qquad m = \frac{5}{2}+\frac{3}{\gamma_2}$$

where x is the diffusion distance, σ is the standard deviation of the diffusion pdf, γ_2 is the excess kurtosis of the distribution, Γ is the gamma function and B the Euler Beta function. In the limit of the excess kurtosis approaching zero, both distributions converge to the normal distribution. Since the diffusion weighted images (DWI's) are recorded in Q-space, the real valued characteristic functions of these distributions model the magnitude of the DWI's,

$$\varphi_{P_{II}}(b) = 2^{\frac{1}{2}-m}(bs)^{m-\frac{1}{2}}J_{\frac{1}{2}(1-2\,m)}(bs)\Gamma\left(\frac{3}{2}-m\right)\,, \qquad \varphi_{P_{VII}}(b) = \frac{2^{\frac{2}{2}-m}(b\alpha)^{m-\frac{1}{2}}K_{m-\frac{1}{2}}(b\alpha)}{\Gamma\left(m-\frac{1}{2}\right)}\,,$$

where J is the Bessel function of the first kind and K the modified Bessel function of the second kind. See figure 2 for an example of these characteristic functions for some values of the excess kurtosis K.

Discussion

This new magnitude model based on the Pearson set of distributions clearly is more realistic for large b-values, since the predicted magnitude of the diffusion weighted image does not increase to arbitrary large values when the kurtosis is positive. The differences between the signals with different kurtosis values are largest for normalized b-values between 1 and 2, as can be seen in the figures 1 and 2. Therefore, to efficiently estimate the kurtosis, b-values in this range should be selected. However, as can be seen in figure 1, the magnitude of the DWI predicted with the traditional model is clearly inaccurate in this range, when the kurtosis is positive. This inaccurate magnitude prediction might lead to a bias in the kurtosis estimates, which might be reduced by using lower b-values. However, the differences between the signals at these lower b-values are very small, which means that very high SNR's will be needed. The new Pearson distribution model of the magnitudes of the DWIs does predict realistic values for any kurtosis and b-value. Therefore, the kurtosis estimated with this model will have a reduced bias.

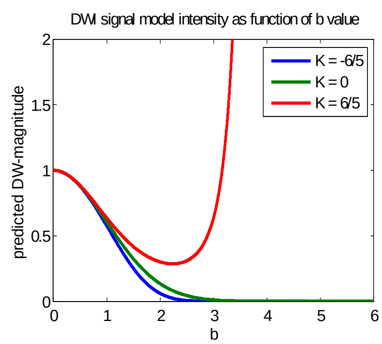

Figure 1: The normalized magnitude of the DWI predicted with the standard diffusion kurtosis model, for several kurtosis values and with normalized b-value.

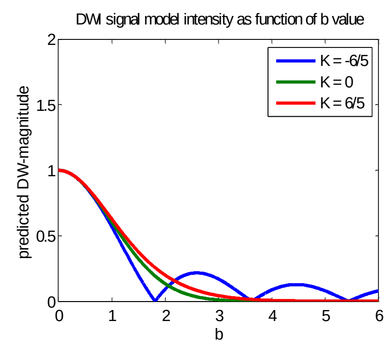

Figure 2: The normalized magnitude of the DWI predicted with the Pearson distribution diffusion kurtosis model, for several kurtosis values and with normalized b-value.

Furthermore, when the gradient strength of the DWIs is optimized based on the predictions of the DW magnitude, the traditional model with unrealistically high predicted magnitudes, which are extremely sensitive to changes in the kurtosis value, will strongly influence the selected gradient strengths. The lower and realistic DW magnitudes predicted with Pearson distribution will be much better for optimizing the gradient strengths.

References

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